Solution, Exam December 16, 2010

1. (a)
$$w = \frac{1}{2}\begin{bmatrix}0.09 & -0.027\\ -0.027 & 0.09\end{bmatrix}^{-1}\begin{bmatrix}0.05\\ -0.01\end{bmatrix} = \frac{1}{2}\begin{bmatrix}12.21 & 3.663\\ 3.663 & 12.21\end{bmatrix}\begin{bmatrix}0.05\\ -0.01\end{bmatrix} = \begin{bmatrix}0.2869\\ 0.0205\end{bmatrix}$$

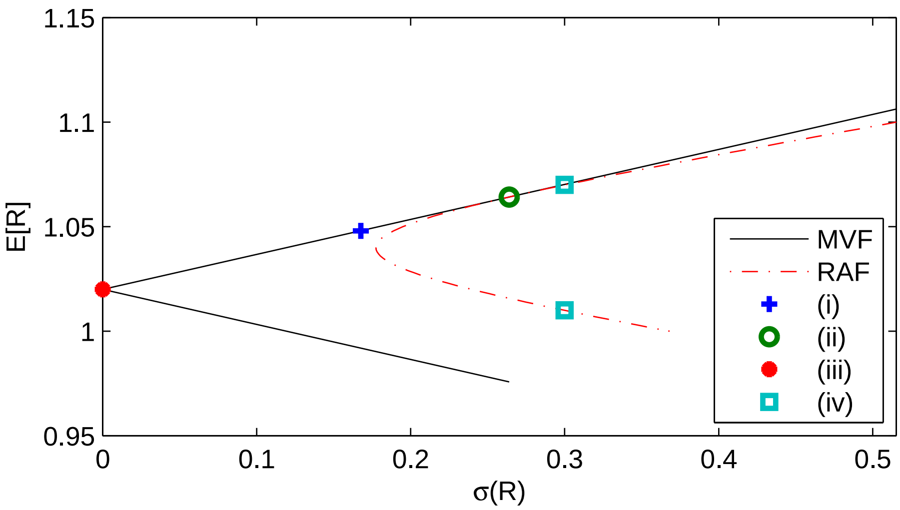
   (b) It's negative covariance with the return to the first asset means it works as a hedge against bad returns to the first asset.

   (c)
$$E[R^p] = 1.02 + (0.2869)(0.05) + (0.0305)(-0.01) = 1.034$$
$$\sigma(R^p) = \sqrt{w'\Omega w} = 0.0838$$

   (d) Alternative 1 (Using direct scaling): Excess return and standard deviation of tangency portfolio is $1/(0.2869 + 0.0305) = 3.15$ times higher than that of the investors portfolio:
$$E[R^{mv}] = 1.02 + 3.15(0.014) = 1.0620$$
$$\sigma(R^{mv}) = 3.15(0.0838) = 0.2639$$

   Alternative 2 (Finding weights of tangency portfolio first):
$$w^{mv} = \frac{1}{0.2869 + 0.0305}\begin{bmatrix}0.2869 & 0.0305\end{bmatrix}' = \begin{bmatrix}0.9039 & 0.0961\end{bmatrix}'$$
$$E[R^{mv}] = (0.9039)(1.07) + (0.0961)(1.01) = 1.0642$$
$$\sigma(R^{mv}) = \sqrt{w'\Omega w} = \sqrt{0.0697} = 0.2640$$

   (e)



   (f)
$$E[R^m] = 1.04$$
$$\sigma^2(R^m) = 0.0315$$
$$\begin{aligned}\text{cov}\left(R^1, R^m\right) &= \text{cov}\left(R^1, 0.5R^1 + 0.5R^2\right)\\ &= 0.5\text{cov}\left(R^1, R^1\right) + 0.5\text{cov}\left(R^1, R^2\right) = 0.5(0.09) + 0.5(-0.027) = 0.0315 = \text{cov}\left(R^2, R^m\right)\end{aligned}$$
$$\beta_1 = \beta_2 = 1$$
$$E[R^1] = 1.02 + \beta_1(0.02) = 1.04 = E[R^2]$$

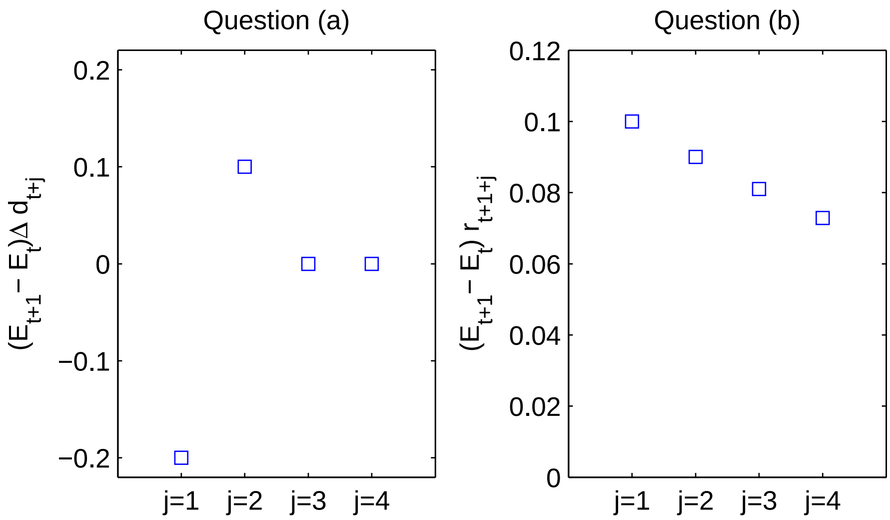
2. (a) From top nodes to bottom nodes:
$$m_{t+1} = \begin{bmatrix}0.9048\\ 1\end{bmatrix} \quad m_{t+2} = \begin{bmatrix}0.9068\\ 0.9975\\ 0.9025\\ 1.0028\end{bmatrix}$$

(b) From top to bottom

$$P_{t+1} = \begin{bmatrix} (0.8)(0.9068)(120) + (0.2)(0.9975)(100) \\ (0.5)(0.9025)(100) + (0.5)(1.0028)(80) \end{bmatrix} = \begin{bmatrix} 107 \\ 85.23 \end{bmatrix}$$
$$P_t = (0.8)(0.9048)(107 + 5) + (0.2)(1)(85.23 + 5) = 99.11$$

(c) Since the price (and hence the realized return) is higher in the up-nodes (when consumption is high), holding equity increases consumption risk. Investors will only be willing to hold it if it has an expected return that is higher than the risk-free rate.

3.



(c)

$$(i): \quad -0.2 + 0.95(0.1) = -.105$$
$$(ii): \quad \frac{0.95}{1-(0.95)(0.9)}0.1 = 0.655$$

See last page of the solution for a detailed explanation of the dividend growth news part.

4. (a) $VR_1 = VR_2 = 1$.

(b) Not at all. This is exactly the setting analyzed by Samuelson. Both expected log returns and variances scale linearly with time, so the risk-return tradeoff is the same at all investment horizons.

(c) Notice that the gross return on equity would be given by $R_{t+1} = \exp(0.1 - 0.4) = 0.74$. While the gross return on the risk-free bonds is $R^f = 1.0513$. If she has initial wealth W_t and allocates 0.5 to equity. The value of her equity is going to be $(0.74)(0.5W_t) = 0.37W_t$, while the value of her risk-free investment is going to be $1.05(0.5)W_t = 0.53W_t$. Her new wealth is the sum of the value of her equity and risk-free assets, or $W_{t+1} = 0.9W_t$. To find the portfolio weight before rebalancing, we need to scale the value of her equity position to her total wealth, or:

$$\omega = \frac{0.53W_t}{0.9W_t} = 0.41$$

(d) She needs to move 9 % of her wealth from risk-free to equity. (Since she currently has (1-0.41) = 0.59 % of her wealth in risk-free assets, this involves selling 0.09/0.59 = 0.15 % of her risk-free bonds and use the proceeds to buy equity.

5. (a) By the law of iterated expectations

$$\begin{aligned} E_t[m_{t+1}|S_t = 1] &= E_t[m_{t+1}|S_{t+1} = 1]\Pr(S_{t+1} = 1|S_t) + E_t[m_{t+1}|S_{t+1} = 2]\Pr(S_{t+1} = 1|S_t) \\ &= (0.95)E_t[m_{t+1}|S_{t+1} = 1] + (0.05)E_t[m_{t+1}|S_{t+1} = 2] \end{aligned}$$

(b)

$$\bar{m}_1 = e^{-0.005+(1/2)(0.005)^2} = 0.995$$
$$\bar{m}_1 = e^{0.005+(1/2)(0.005)^2} = 1.005$$

(c)

$$R^f(boom) = \left((0.95)(0.995) + (0.05)(1.005)\right)^{-1} = 1.0045$$
$$R^f(recession) = \left((0.25)(0.995) + (0.75)(1.005)\right)^{-1} = 0.9975$$

(d) Expected consumption growth is higher in the boom state. Because marginal utility of consumption is decreasing in consumption, this will, on average, make the marginal utility of consumption lower tomorrow. If the interest rates does not change, investors want to borrow to spend more today. (When marginal utility is higher.) This drives up interest rates. [*Your answer somehow has to contain the relation between consumption growth and the SDF.*]

1 Detailled explanations for question 3:

This table decomposed the dividend growth rates into news arriving in different periods. The first column gives the effect of the news from $t-1$ on Δd_{t+j}, the second the effect of news from t, etc. ϵ_{t-1} and ϵ_t are old news, so they are already incorporated in the stock price. ϵ_{t+2}, ϵ_{t+3}, etc. are news from the future. They will be incorporated in the stock price some point in the future, but since investors don't know them yet, they have no effect on the return at $t+1$. The "new" news at $t+1$ is what is going to affect the return at $t+1$. I mark them with red. Notice that I've turned around the equation for Δd_{t+1} so that the oldest news are first. (Normally you write the newest news first, but I wanted to write the table in a way that is consistent with time flowing from left to right.)

Δd_{t+j}	Timing of information						News at t+1	Effect of news on r_{t+1}
Δd_{t+1}	$\frac{1}{5}\epsilon_{t-1}$	$+\frac{1}{2}\epsilon_t$	$+\epsilon_{t+1}$				ϵ_{t+1}	ϵ_{t+1}
Δd_{t+2}		$\frac{1}{5}\epsilon_t$	$+\frac{1}{2}\epsilon_{t+1}$	$+\epsilon_{t+2}$			$\frac{1}{2}\epsilon_{t+1}$	$\rho\frac{1}{2}\epsilon_{t+1}$
Δd_{t+3}			$\frac{1}{5}\epsilon_{t+1}$	$+\frac{1}{2}\epsilon_{t+2}$	$+\epsilon_{t+3}$		$\frac{1}{5}\epsilon_{t+1}$	$\rho^2\frac{1}{5}\epsilon_{t+1}$
Δd_{t+4}				$\frac{1}{5}\epsilon_{t+2}$	$+\frac{1}{2}\epsilon_{t+3}$	$+\epsilon_{t+4}$	0	0
$\eta_{d,t+1}$: Total effect of dividend news on r_{t+1}.								$(1+\frac{1}{2}\rho+\frac{1}{5}\rho^2)\epsilon_{t+1}$

The effect of the news on Δd_{t+1} has an immediate effect on cash-flows, so there's no ρ in front in the last row, dividend growth at $t+2$ is only going to affect cash-flows from next period on, so you "discount" the effect by multiplying with ρ, and dividend growth at $t+3$ is only going to affect cash-flows from 2 periods in the future onwards, so you discount that effect by multiplying with ρ^2. You get the total effect on returns by summing up the elements in the last column.