Faculty of Economics and Business Administration

Exam:	Investments 4.1
Code:	60412040
Coordinator:	Frode Brevik
Date:	October 29, 2010
Time:	8:45
Duration:	2 hours and 45 minutes
Calculator allowed:	Yes
Graphical calculator allowed:	Yes
Number of questions:	20 part questions (numbered (a), (b), (c))
Type of questions:	Open
Answer in:	English
Credit score:	Each part question give 0.5 of the total grade. Some part questions are divided into subparts numbered with (i), (ii), (iii)
Grades:	Final grades will be made public by November 12.
Inspection:	Monday, November 8, 2010 at 18.00. Room to be announced.
Number of pages:	5 (including front page and formula sheet)

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between three stocks and risk free bonds. The returns to the three stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. Expected returns are given by

$$R^f = 1.02 \qquad E_t[R_{t+1}] = \begin{bmatrix} 1.12 \\ 1.07 \\ 1.02 \end{bmatrix}$$

The return of each risky asset has a standard deviation of 50 % and the returns to the three stocks are uncorrelated. The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$E[R^p_{t+1}] - \frac{\lambda}{2}\sigma^2(R^p_{t+1})$$

where $E[R^p_{t+1}]$ and $\sigma^2(R^p_{t+1})$ are the expected return and the variance of the return to the investors portfolio.

(a) Show that the optimal portfolio weights w are given by

$$w = \frac{1}{\lambda}\Omega^{-1}(E_t[R_{t+1}] - R^f \iota)$$

(b) i. What percentage should an investor with $\lambda = 1$ allocate to risk-free bonds?

ii. How does the allocation to risk-free bonds change as λ increases. (Argue why in terms of the investors preferences.)

(c) Compute the expected return and standard deviation of the investor's portfolio.

(d) Compute the expected return and variance of the return to the tangency portfolio between the risky-asset frontier and the mean-variance frontier.

(e) Sketch the Risky-asset frontier and the mean-variance frontier in the standard standard-deviation/expected returns diagram. Make sure to label the axis and mark the points corresponding to:

i. The Investor's portfolio.

ii. The tangency portfolio.

iii. Risk-free bonds.

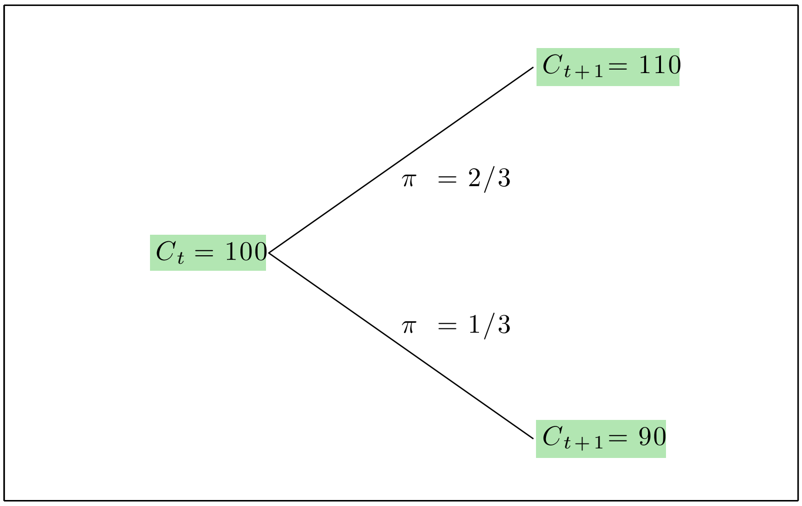
iv. The three stocks.

(f) Assume that market capitalization of the three stocks are the same ($P_1N_1 = P_2N_2 = P_3N_3$, where P_i and N_i is the share price and the number of shares outstanding for stock i, respectively.) Which condition that we used to derive the CAPM would be violated?

2. Consider the following simple economy. The probability of going to the up state next period is equal to 2/3. Assume the representative investor has a utility function given by

$$U(C) = \ln C,$$

and he has a time discount factor of $\theta = 0.95$. Investors can trade freely in Arrow-Debreu securities for both states of the world next period.



(a) Assume that the values for C_t and C_{t+1} given in the figure are the equilibrium consumption levels of the representative investor. Find the implied values of:

i. The stochastic discount factor between time t and $t+1$ for both possible states of the economy.

ii. The price of an Arrow-Debreu security that pays out 1 in the upstate and the price of an Arrow-Debreu security that pays out 1 in the downstate.

iii. The price at t of a security that pays out 1 in both possible states of the economy at time $t+1$.

(b) Find the risk-neutral probabilities of each state.

(c) Another investor, also with a log utility function, has the choice between two jobs, one that pays a salary of 50 in both states of the economy and one that pays out a salary of 100 in the up state and 0 in the down state. This investor can also trade freely in Arrow securities at the prices you found in part (a).

i. Which job offer should the investor accept.

ii. Could your answer to this question change if the investor was very risk-averse? Argue in one sentence why/why not.

3. Campbell and Shiller decompose unexpected stock returns into

$$r_{t+1} - E_t[r_{t+1}] = (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r_{t+1+j}$$

Assume $\rho = 0.95$. Let log dividend growth be given by

$$\Delta d_{t+1} = \epsilon_{t+1} + 0.5\epsilon_t + 0.2\epsilon_{t-1}, \qquad \epsilon \sim I.I.D, \mathcal{N}(0, (0.1)^2)$$

(a) Sketch how expected log dividend growth rates (Δd_{t+j}) change from time t to $t+1$ in response to $\epsilon_{t+1} = -0.2$ for $j = \{1, 2, 3, 4\}$

(b) Compute the effect of this news about future dividend growth on r_{t+1}.

(c) Assume that expected future returns are constant, so that $E_{t+1}[r_{t+1+j}] = E_t[r_{t+1+j}], \quad j \geq 1$. Find the standard deviation of returns implied by the assumed process for dividend growth.

4. (Pricing an American Option) The current price of a stock is 100 and the standard deviation of its annual log return is 0.5. The continuously compounded annual risk-free rate is $r^f = 0.04$. In 3 months the company will pay a dividend of 5. For the rest of the exercise, use a binomial tree with nodes 3 months apart that goes from the current period to 6 months into the future.

(a) Give the numbers for the factors u and d, and the risk-neutral probability $\tilde{\pi}$ of going to the "up-node".

(b) Sketch the tree and fill in stock-prices at every node of the tree. For the nodes at time $t+0.25$, use the cum-dividend price. (The price before dividends are paid).

(c) Use backwards induction to price at each node an American call option with strike price 110 which matures in 6 months. Remember to check for whether early exercise is optimal.

5. Assume that interest rates are time-varying, but expected excess log returns to all risky assets are constant over time. Investor can invest in three risky assets, stocks of company 1, stocks of company 2, and gold. The log return to stock i is denoted by r^i, while the log return to gold is denoted r^g All log returns are conditionally normal with a constant covariance matrix Ω. The investment opportunity set, as a function of the log risk-free rate r_t^f is given by:

$$\mu = \begin{bmatrix} E[r_{t+1}^1] + \sigma_1^2 \\ E[r_{t+1}^2] + \sigma_2^2 \\ E[r_{t+1}^g] + \sigma_g^2 \end{bmatrix} = r_t^f + \begin{bmatrix} 0.05 \\ 0.03 \\ -0.01 \end{bmatrix} \quad \Omega = \begin{bmatrix} (1/2)^2 & 0 & 0 \\ 0 & (1/3)^2 & 0 \\ 0 & 0 & (1/4)^2 \end{bmatrix}$$

The current risk free rate is given by

$$r_t^f = 0.05.$$

The realized return to the two stocks are uncorrelated with changes in the interest rate, while the return to gold has a negative covariance with changes to the risk-free rate of -0.001.

(a) Assume the representative investor has a time invariant value function given by:

$$V(W_t, r_t^f) = k + \frac{1}{1-\gamma}(1 + r_t^f)^{1-\gamma} W_t^{1-\gamma}$$

with $\gamma = 2$. Find the investors optimal portfolio using the general formula

$$w = \left(-\frac{V_w}{V_{ww}W}\right)\Omega^{-1}(\mu - r_f) + \left(-\frac{V_{ws}}{V_{ww}W}\right)\Omega^{-1}\Phi$$

(b) One can think of the investor as holding a combination of three funds: (i) risk-free bonds, (ii) a mean-variance efficient portfolio, (iii) a portfolio which provides the best hedge against changes in the investment opportunity set. Find the portfolio weights of the last two portfolios.

(c) Assume that, by chance, the realized log return on the second stock and gold between t and $t+1$ were equal to the risk-free rate at time t, but that the realized log return on the first stock was $r_{t+1} = -0.5$. How does the investor need to rebalance his portfolio in order to maintain an optimal allocation to each asset if the risk-free rate stays the same, so $r_{t+1}^f = r_t^f$. (A verbal answer is sufficient, I don't expect you to provide numbers.)

6. Assume you have the following data on annual log forward rates:

$$f_t^{(0)} = 0.03$$
$$f_t^{(1)} = 0.04$$
$$f_t^{(2)} = 0.05$$

where $f_t^{(i)}$ is the rate at which you can borrow (or lend) money from $t+i$ to $t+i+1$.

(a) Find the 1, 2, and 3 year log spot rates.

(b) Find the log prices of discount bonds maturing in 1, 2, and 3 years.

Important formulas

Vector derivatives

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x$$
$$\frac{\partial x'Ax}{\partial x} = (A + A')x$$

Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & 0 & \dots & 0 \\ 0 & a_{2,2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{n,n} \end{bmatrix}^{-1} = \begin{bmatrix} 1/a_{1,1} & 0 & \dots & 0 \\ 0 & 1/a_{2,2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & 1/a_{n,n} \end{bmatrix} \qquad \text{(Diagonal matrix)}$$

Expectations, variances

$$E_t[ax + by] = aE_t[x] + bE_t[y] \qquad \text{(Linearity)}$$
$$E_t[E_{t+1}[x]] = E_t[x] \qquad \text{(L.I.E.)}$$
$$\text{var}\,(x) = E[x^2] - E[x]^2$$
$$\text{var}\,(ax) = a^2\text{var}\,(x)$$
$$\text{cov}\,(x, y) = E[x\ y] - E[x]E[y]$$
$$\text{cov}\,(ax, y) = a\text{cov}\,(x, y) = \text{cov}\,(x, ay)$$

Expectation of a log-normal

$$E[e^x] = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{if} \qquad x \sim \mathcal{N}(\mu, \sigma^2)$$

MA(q) processes

If

$$x_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \dots + \theta_q\epsilon_{t-q} \qquad \text{with } \epsilon_t \text{ I.I.D.}$$

Then

$$\text{var}\,(x_t) = (1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2)\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = (\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \dots + \theta_{q-j}\theta_q)\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \frac{\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \dots + \theta_{q-j}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \dots + \theta_q^2} \qquad \text{(auto-correlation)}$$

AR(1) process

If

$$x_t = \mu + \rho x_{t-1} + \epsilon_t \qquad \text{with } |\rho| < 1, \epsilon_t \sim \mathcal{N}(0, \sigma^2),\ \epsilon_t\ I.I.D.$$

Then

$$\text{var}\,(x_t) = \frac{1}{1 - \rho^2}\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = \frac{\rho^j}{1 - \rho^2}\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \rho^j \qquad \text{(auto-correlation)}$$

VR_k statistic

$$VR_k = \frac{1}{k}\frac{\text{var}\,(r_{t,t+k})}{\text{var}\,(r_{t+1})} = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\phi_j,$$

where ϕ_j is the jth autocorrelation coefficient of returns.