1 Sample exam solution, Investments 4.1, October 29, 2010

1. (a) The F.O.C. of the investor is

$$\begin{aligned} 0 &= \frac{d}{dw}\left(w'(E[R]-R^f)+R^f-\frac{\lambda}{2}w'\Omega w\right) \\ &= (E[R]-R^f)-\frac{\lambda}{2}2\Omega w \\ &= (E[R]-R^f)-\lambda\Omega w \\ \Rightarrow \lambda\Omega w &= (E[R]-R^f) \end{aligned}$$

Premultiplying both sides with $(1/\lambda)\Omega^{-1}$

$$w = \frac{1}{\lambda}\Omega^{-1}(E_t[R_{t+1}]-R^f\iota)$$

(b) i. The optimal portfolio weights for the risky assets are

$$w = 4\begin{bmatrix}0.1\\0.05\\0\end{bmatrix} = \begin{bmatrix}0.4\\0.2\\0\end{bmatrix}$$

So he should allocate $1-0.6=0.4$, or 40 % to risk-free bonds.

ii. As λ increases the investor allocates less to each of the risky assets and hence more to risk-free bonds. This reflects that with a higher λ the investor has a stronger dislike for return variance. (A higher risk aversion.)

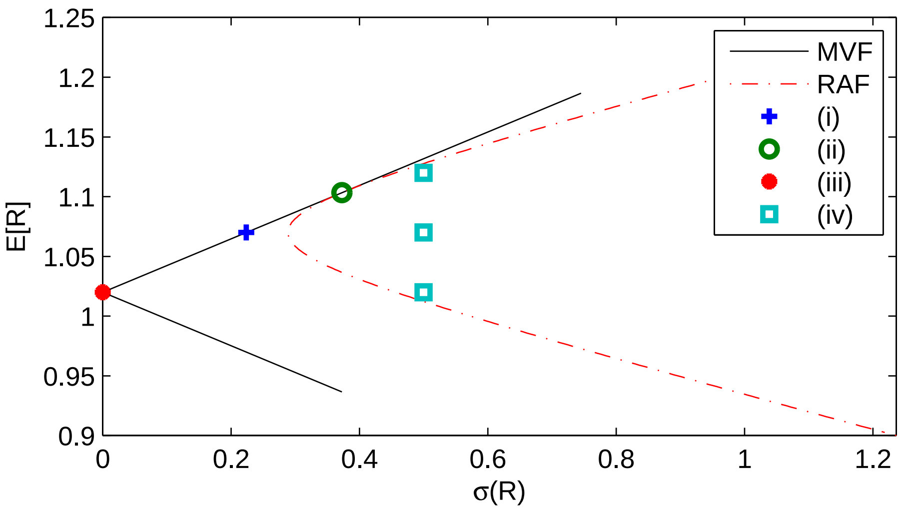
(c)

$$E[R^p] = 1.02 + 0.4(0.1) + 0.2(0.05) = 1.07 \qquad \sigma^2(R^p) = (0.4)^2(1/4) + (0.2)^2(1/4) = 0.05$$

(d)

$$E[R^\star] = (2/3)1.12 + (1/3)1.07 = 1.1033 \qquad \sigma^2(R^\star) = (2/3)^2(1/4) + (1/3)^2(1/4) = 0.1389$$

(e) The labels (i) to (iv) refer to the part question answered.



(f) The tangency portfolio has different weights than the market portfolio:

$$w^\star = \begin{bmatrix}2/3\\1/3\\0\end{bmatrix} \neq \begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix} = w^m$$

(Instead of the CAPM, you would have: $E[R^i] = R^f + \beta^{\star,i}(E[R^\star - R^f)$ where $\beta^{\star,i}$ is the beta of the ith stock with the tangency portfolio. This follows by the derivation we did in class, and this is always true. What's missing is that we cannot replace $R^\star$ with R^m unless they are really the return on the same portfolio.)

2. (a) i.

$$m_{t+1} = \theta \frac{U'(C_{t+1})}{U'(C_t)} = \begin{bmatrix} (0.95)(1/110)/(1/100) \\ (0.95)(1/90)/(1/100) \end{bmatrix} = \begin{bmatrix} 0.8636 \\ 1.0556 \end{bmatrix}$$

ii. Using $q_i = \pi_i m_i$, we get:

$$q = \begin{bmatrix} 0.5758 \\ 0.3519 \end{bmatrix}$$

iii.

$$B = \sum q = 0.9276$$

(b) *Quickest way:*

$$\tilde{\pi} = q / \sum q = \begin{bmatrix} 0.6207 \\ 0.3793 \end{bmatrix}$$

Less direct alternatives:

i. Use links between SDF and risk-neutral probabilities:

$$\tilde{\pi}_i = \pi \frac{m_i}{E_t[m_{t+1}]} = \pi_i \frac{m_i}{B}$$

ii. Use the fact that the expected return under the risk-neutral probability measure is equal to the risk-free rate together with the price and payoff of one of the AD securities. E.g. with security 1:

$$R^f = \frac{1}{B} = \tilde{\pi}_1 \frac{1}{q_1} + (1 - \tilde{\pi}_1) \frac{0}{q_1} = \frac{\tilde{\pi}_1}{q_1} \qquad \Rightarrow \qquad \tilde{\pi}_1 = \frac{q_1}{B}, \quad \tilde{\pi}_2 = 1 - \tilde{\pi}_1$$

Naturally, you need to use the fact that probabilities sum to 1.

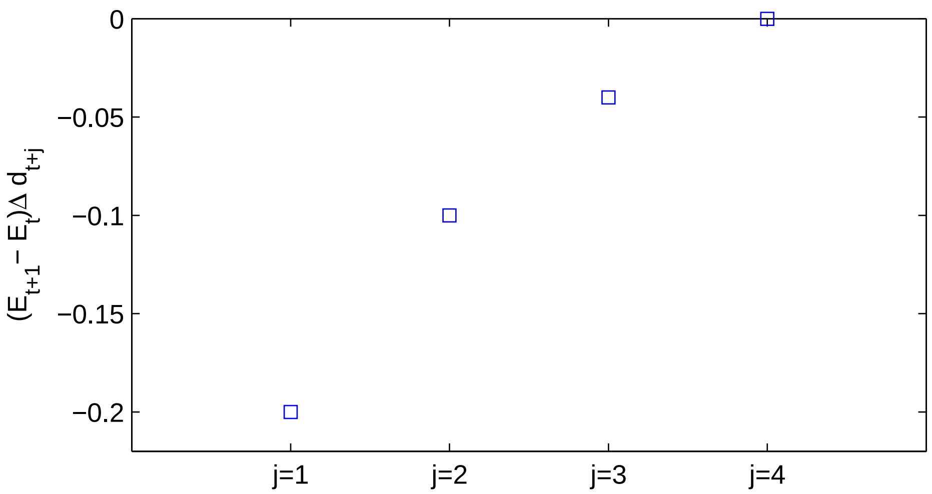
(c) *Most natural solution, corresponding to your assignment 2, view jobs as a portfolio of AD securities:*

i. should accept the second offer, because $(100)(q_1) = 57.58 > (50)(B) = 46.38$

ii. No, the investor only cares about the present value of the income. He can offset any risk by taking the appropriate position in Arrow securities.

Alternatively, use the risk-neutral probabilities from (b) and find that job 2 has a higher expected earning under this probability measure.

3. (a) Impulse-response:



(b)

$$r_{t+1} - E_t[r_{t+1}] = \left(1 + (0.5)(0.95) + (0.2)(0.95)^2\right)(-0.2) = -0.3311$$

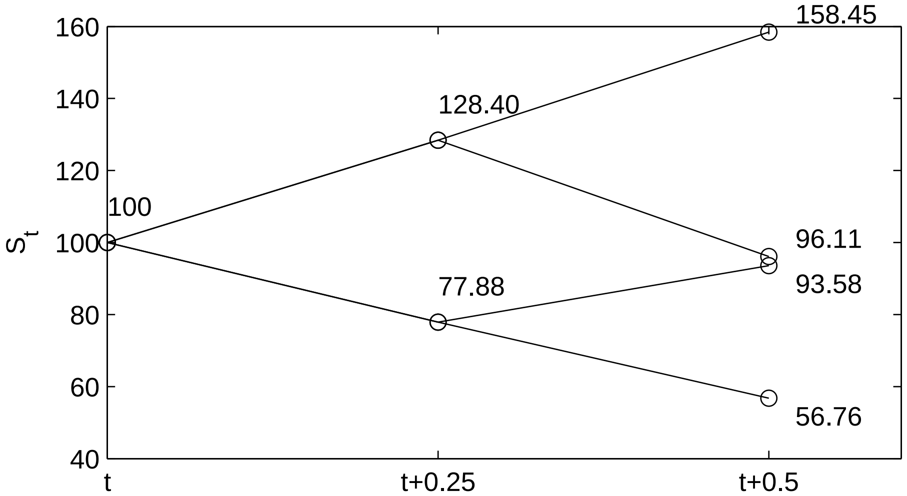
(c)

$$\sigma(r_{t+1}) = \left(1 + (0.5)(0.95) + (0.2)(0.95)^2\right)\sigma(\epsilon) = 0.1656$$

4. (a)

$$u = \exp(0.5\sqrt{0.25}) = \exp(0.5) = 1.2840 \qquad d = \exp(-0.5) = 0.7788 \qquad \tilde{\pi} = \frac{e^{0.01} - d}{u - d} = 0.4577$$

(b) Binomial tree for the stock price:



(c) At $t + 0.5$ the option is out of the money at all nodes except the last where it's inner value is 48.45. At the upper node at time $t + 0.25$, the inner value of the option (18.40) is lower than the expected discounted value of the option at $t + 0.5$. $e^{-0.01}\tilde{\pi}48.45 = 21.96$), so the value in the up node at time $t + 0.25$ is 21.96. In the down-node, the inner value of the option is negative, and it has an expected value of zero at $t + 0.5$, so the option is valueless. At time t, the option has zero inner value, but an expected value of $e^{-0.01}\tilde{\pi}21.96 = 9.95$, which is then the option value at t.

5. (a) Preliminary calculations:

$$-\frac{V_w}{V_{ww}W} = -\frac{(1+r_t^f)^{1-\gamma}W_t^{-\gamma}}{-\gamma(1+r_t^f)^{1-\gamma}W_t^{-\gamma-1}W_t} = \frac{1}{\gamma}$$

$$-\frac{V_{ws}}{V_{ww}W} = -\frac{(1-\gamma)(1+r_t^f)^{-\gamma}W_t^{-\gamma}}{-\gamma(1+r_t^f)^{1-\gamma}W_t^{-\gamma}} = -\frac{\gamma-1}{\gamma}\frac{1}{(1+r_t^f)}$$

$$w = \frac{1}{2}\begin{bmatrix}4 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}0.05\\ 0.03\\ -0.01\end{bmatrix} + \frac{(1-2)}{(2)(1.05)}\begin{bmatrix}4 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 16\end{bmatrix}\begin{bmatrix}0\\ 0\\ -0.01\end{bmatrix} = \begin{bmatrix}0.100\\ 0.135\\ -0.080\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0.0076\end{bmatrix} = \begin{bmatrix}0.1\\ 0.135\\ -0.07\end{bmatrix}$$

(b) Scaling the vectors for myopic and hedging demand to sum to unity gives:

$$w^{mv} = \frac{1}{0.1550}\begin{bmatrix}0.100\\ 0.135\\ -0.080\end{bmatrix} = \begin{bmatrix}0.6452\\ 0.8710\\ -0.5161\end{bmatrix} \qquad w^h = \frac{1}{0.0076}\begin{bmatrix}0\\ 0\\ 0.0076\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

(c) The investor needs to shift his portfolio towards the first stock. This would mean selling all other assets.[1]

[1] Only 1/3 of you got this point, even though I promised you that I would ask just this question. Another thing I noticed on this point is that many of you have problems visualizing what happens to portfolio weights after returns are realized. Normalizing the wealth at time t to 1, the new value of the ith asset in your portfolio before rebalancing is $e^{r^i_{t+t}}w_i$. So, unless the returns on each asset are equal, you need to rebalance. This is also true if realized returns on each asset equal the expected return on that asset. This is why I say that the realized returns on the other two asset are equal to the risk-free rate and not that they are equal to their expected values.

6. (a)

$$r_t^{(1)} = f_t^{(0)} = 0.03$$
$$r_t^{(2)} = \frac{1}{2}(f_t^{(0)} + f_t^{(1)}) = 0.035$$
$$r_t^{(3)} = \frac{1}{3}(f_t^{(0)} + f_t^{(1)} + f_t^{(2)}) = 0.04$$

(b)

$$b_t^{(1)} = -r_t^{(1)} = -0.03$$
$$b_t^{(2)} = -2r_t^{(2)} = -0.07$$
$$b_t^{(3)} = -3r_t^{(3)} = -0.12$$