Investments 4.1
Course code: 60412040

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Date: December 10, 2009
Time: 15:15 – 18:00
(2 hours, 45 minutes)

Parts:	The exam contains 20 parts. Some parts are divided into subparts. The subparts are numbered by (i), (ii), etc.
Grading:	Each part count equally to the exam grade.
Results:	Results will be posted on Blackboard at the latest by Thursday, December 17.
Inspection:	You can inspect your marked exam papers Thursday, December 17, 16:15. The room will be announced.
Remark:	Provide complete answers, including computations where appropriate. On verbal questions: always provide motivation/explanation of your answer in terms of the economic mechanism at play. A short "yes" or "no" will never do as an answer. But be concise in your answers, otherwise you'll loose too much time writing them down.

Notice formula sheet at the end!

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between two stocks and risk free bonds. The returns to the two stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. Expected returns are given by

$$R^f = 1.02 \qquad E_t[R_{t+1}] = \begin{bmatrix} 1.12 \\ 1.06 \end{bmatrix}$$

The return of each risky asset has a standard deviation of 50 % and the returns to the two assets are uncorrelated. The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$w' E_t[R_{t+1}] + (1 - w'\iota) R^f - \frac{\lambda}{2} w' \Omega w$$

(a) The optimal portfolio weights w for the risky assets are given by

$$\frac{1}{\lambda} \Omega^{-1} (E_t[R_{t+1}] - R^f \iota)$$

i. What percentage should an investor with $\lambda = 1$ allocate to risk free bonds?
ii. How does the allocation to risk-free bonds change as λ increases. (Argue why in terms of the investors preferences.)

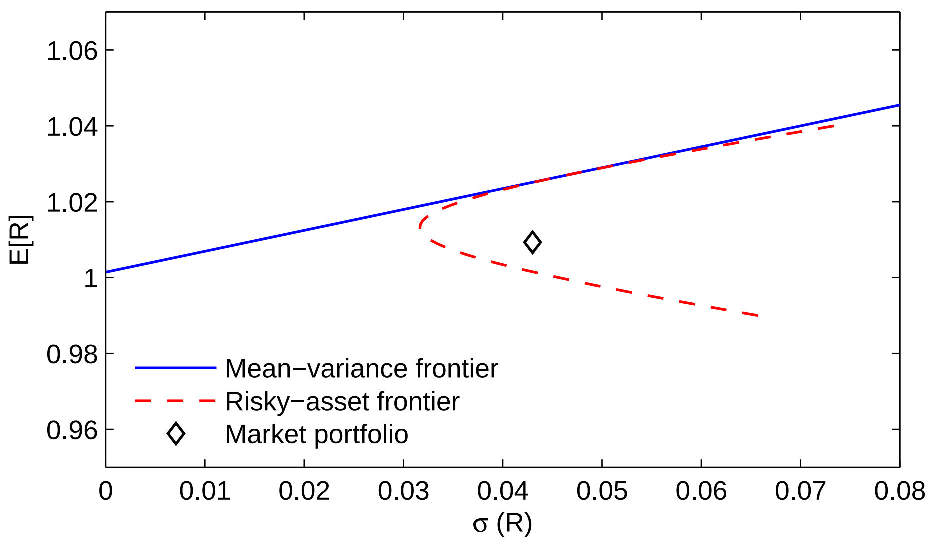
(b) Compute the expected return and standard deviation of the investor's portfolio.

(c) Compute the mean and variance of the tangency portfolio between the risky-asset frontier and the mean-variance frontier.

(d) Sketch the Risky-asset frontier and the mean-variance frontier in the standard standard-deviation/expected returns diagram. Make sure to label the axis and mark the points corresponding to:

i. The Investor's portfolio.
ii. The tangency portfolio.
iii. Risk-free bonds.
iv. The two stocks.

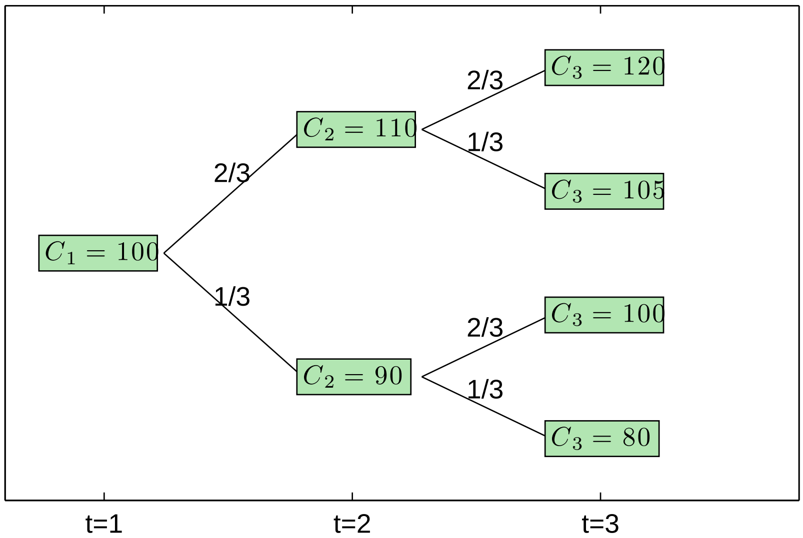
2. This figure is taken from your first assignment. The risky-asset frontier is computed by assuming that the expected return to each of the risky assets available are equal to their historical averages. The point marked in the figure gives the expected return and the standard deviation of the market portfolio.



(a) Is the situation given in the figure compatible with the CAPM? (Could the CAPM hold here?) Justify your answer.

(b) Copy the figure in your answer and sketch how the figure would change in equilibrium if all investors were mean-variance optimizers.

3. Consider the following simple economy. The probability of going up is equal to 2/3 always. P_t is the price of a stock in year t and C_t is consumption in year t, both depending on the state of the economy. There are no dividends at any points in the tree.



(a) Assume the representative investor has a period utility function given by

$$U(C) = \ln C$$

And a time discount factor of $\theta = 1$. Find the discount factor m_{t+1} at each node in the two years $t = 2$ and $t = 3$.

(b) Use your results to find

i. the price at $t = 1$ of a risk-free zero-coupon bond that pays out 1 at time $t = 2$.

ii. the price in each of the two states at time $t = 2$ of a risk-free zero-coupon bond that pays out 1 at time $t = 3$.

iii. the price at time $t = 1$ of a risk-free zero-coupon bond that pays out 1 at time $t = 3$.

(c) Use the prices of the two zero coupon bonds at time $t = 1$ to find the 1 and 2 year spot rate.

(d) $V(t, P_t)$ is the value of an American put option with strike price $K = 100$ which matures at $t = 3$. Find the value of the option at each node of the tree. (Remember to check for early exercise.)[1]

4. The log return to a particular stock is given by

$$r_{t+1} = 0.05 + \epsilon_{t+1} - 0.1\epsilon_t \qquad \epsilon_{t+1} = \mathcal{N}(0, 0.2^2) \tag{1}$$

and ϵ_t I.I.D.

(a) Compute

$$\frac{E[r_{t+1}]}{\text{var}\,(r_{t+1})} \qquad \text{and} \qquad \frac{E[r_{t,t+2}]}{\text{var}\,(r_{t,t+2})}$$

As well as the VR_1 and VR_2 statistics.

(b) How would you expect the average static allocation to equity differ between a 1 year and a 2 year investors based on your result above. Provide an economic explanation. (Assume both investors have a power utility function.)

[1] A put option is a right to sell a share for a particular price.

(c) Assume $\epsilon_{t-1} = 0$, $\epsilon_t = 0$ and $\epsilon_{t+1} = 0.2$. Should a power utility investor with a 1 year investment horizon allocate more or less to equity at time $t+1$ than she did at time t? Provide an argument in terms of the risk-return trade-off the investor faces at t and $t+1$.

5. Assume that excess log return to equity is given by:

$$r^e_{t+1} = r_{t+1} - r^f = 0.03 + 0.2dp_t + \epsilon_{t+1}, \qquad \epsilon_{t+1} = \mathcal{N}(0, 0.2^2),$$

where dp_t is the log dividend yield at time t. The conditional covariance of the dividend yield with excess equity returns is given by

$$\text{cov}_t\left(r^e_{t+1}, dp_{t+1}\right) = -0.001$$

An investor's value function at time t is given by

$$V(t, W, dp_t) = (dp_t)^{1-\gamma} \frac{W_t^{1-\gamma}}{1-\gamma}$$

(a) Give a possible economic reason why the value function of the investor depend on the current log dividend yield.

The optimal total equity allocation of the investor is given by:

$$w = \left(-\frac{V_W}{V_{WW}W}\right) \frac{E_t[r^e_{t+1}]}{\text{var}_t\left(r^e\right)} + \left(-\frac{V_W}{V_{WW}W}\right) \frac{V_{WS}\ \text{cov}_t\left(r^e, S\right)}{V_W\ \text{var}_t\left(r^e\right)}$$

Assume $dp_t = 0.1$.

(a) Compute the myopic demand of the investor for $\gamma = 4$.

(b) Compute the hedging demand of the investor for $\gamma = 4$.

Important formulas

Vector derivatives

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x$$
$$\frac{\partial x'Ax}{\partial x} = (A + A')x$$

Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \qquad \text{(Special case)}$$

Expectations, variances

$$E_t[a + bx] = a + bE[x] \qquad \text{(Linearity)}$$
$$E_t[E_{t+1}[x]] = E_t[x] \qquad \text{(L.I.E.)}$$
$$\text{var}\,(x) = E[x^2] - E[x]^2$$
$$\text{var}\,(ax) = a^2 \text{var}\,(x)$$
$$\text{cov}\,(x, y) = E[x\ y] - E[x]E[y]$$
$$\text{cov}\,(ax, y) = a\text{cov}\,(x, y) = \text{cov}\,(x, ay)$$

Expectation of a log-normal

$$E[e^x] = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{if} \qquad x \sim \mathcal{N}(\mu, \sigma^2)$$

MA(q) processes

If

$$x_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} \qquad \text{with } \epsilon_t \text{ I.I.D.}$$

Then

$$\text{var}\,(x_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = (\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q)\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \frac{\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \qquad \text{(auto-correlation)}$$

AR(1) process

If

$$x_t = \mu + \rho x_{t-1} + \epsilon_t \qquad \text{with } \epsilon_t \sim \mathcal{N}(0, \sigma),\ \epsilon_t\ I.I.D.$$

Then

$$\text{var}\,(x_t) = \frac{1}{1 - \rho^2}\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = \frac{\rho^j}{1 - \rho^2}\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \rho^j \qquad \text{(auto-correlation)}$$

VR_k statistic

$$VR_k = \frac{1}{k}\frac{\text{var}\,(r_{t,t+k})}{\text{var}\,(r_{t+1})} = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\phi_j,$$

where ϕ_j is the jth autocorrelation coefficient of returns.