Investments 4.1
Course code: 60412040

Lecturer: Frode Brevik, fbrevik@feweb.vu.nl, 598-5057

Date: October 22, 2009
Time: 08:45 – 11:30
(2 hours, 45 minutes)

Parts:	The exam contains 20 parts. Some parts are divided into subparts. The subparts are numbered by (i), (ii), etc.
Grading:	Each part will give 3.5 points of the total 70.
Results:	Results will be posted on Blackboard on Friday, October 30.
Inspection:	You can inspect your marked exam papers Thursday, November 5, 17:30. The room will be announced.
Remark:	Provide complete answers, including computations where appropriate. On verbal questions: always provide motivation/explanation of your answer in terms of the economic mechanism at play. A short "yes" or "no" will never do as an answer. But be concise in your answers, otherwise you'll loose too much time writing them down.

Notice formula sheet at the end!

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between two stocks and risk free bonds. The returns to the two stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. Expected returns are given by

$$R^f = 1.01 \qquad E_t[R_{t+1}] = \begin{bmatrix} 1.09 \\ 1.05 \end{bmatrix}$$

The return of each risky asset has a standard deviation of 50 % and the correlation coefficient between their returns is given by $\rho = 0.6$. The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$w' E_t[R_{t+1}] + (1 - w'\iota) R^f - \frac{\lambda}{2} w' \Omega w$$

   (a) Show (using matrix algebra) that the optimal portfolio weights w for the risky assets are given by

$$\frac{1}{\lambda} \Omega^{-1} (E_t[R_{t+1}] - R^f \iota)$$

   (b) Compute the optimal portfolio weights assuming the investor has $\lambda = 1$.

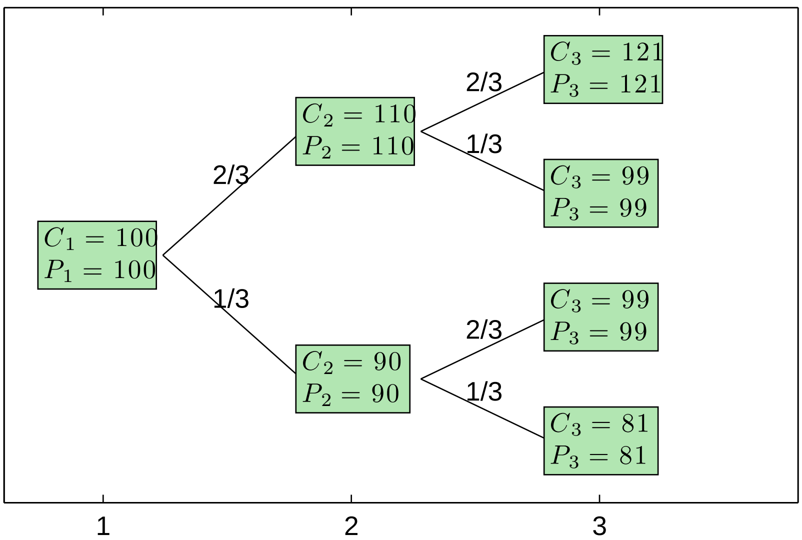
   (c) Compute the expected return and standard deviation of the investor's portfolio.

   (d) Compute the mean and variance of the tangency portfolio between the risky-asset frontier and the mean-variance frontier.

   (e) Sketch the Risky-asset frontier and the mean-variance frontier in the standard standard-deviation/expected returns diagram. Make sure to label the axis and mark the points corresponding to:

      i. The Investor's portfolio.
      ii. The tangency portfolio.
      iii. Risk-free bonds.
      iv. The two stocks.

2. Consider the following simple economy. The probability of going up is equal to 2/3 always. P_t is the price of a stock in year t and C_t is consumption in year t, both depending on the state of the economy. There are no dividends at any points in the tree.



   (a) Assume the representative investor has a period utility function given by

$$U(C) = \ln C$$

   And a time discount factor of $\theta = 1$. Find the discount factor m_{t+1} at each node in the two years $t = 2$ and $t = 3$.

(b) Use your results to find

   i. the price at $t = 1$ of the Arrow-Debreu security that pays out 1 unit of consumption in the highest node at $t = 3$.

   ii. the price at $t = 1$ of the Arrow-Debreu security that pays out 1 unit of consumption in the lowest node at $t = 3$.

   iii. Why do the two prices differ in the way they do. (Make an economic argument.)

(c) $V(t, P_t)$ is the value of an American put option with strike price $K = 100$ which matures at $t = 3$. Find the value of the option at each node of the tree. (Remember to check for early exercise.)[1]

3. In one of your exercises we showed that, if W_t is the current wealth level of an investor who maximizes the expected utility of consumption at time $t+1$ as given by the CARA utility function:

$$U(C_{t+1}) = -e^{-aC_{t+1}},$$

and the returns to the different stocks she can invest in is jointly normally distributed with:

$$R_{t+1} = \mathcal{N}(\mu, \Omega),$$

then her optimal weights to each of the risky securities is given by

$$w = \frac{1}{aW_t}\Omega^{-1}(\mu - R^f\iota)$$

(a) How would the amount the investor invests in each risky security vary with the wealth level of the investor? Is this a plausible description of investor behavior? (Motive your answer.)

(b) If all investors in the economy maximize the same utility function as this investor, why can we be sure the CAPM holds?

4. In your third assignment, you simulated equity returns based on the following VAR for log equity returns and the dividend-yield:

$$\begin{bmatrix} r_{t+1} \\ D_{t+1}/P_{t+1} \end{bmatrix} = \begin{bmatrix} -0.022 \\ 0.0016 \end{bmatrix} + \begin{bmatrix} 0 & 2.039 \\ 0.001587 & 0.9051 \end{bmatrix} \begin{bmatrix} r_t \\ D_t/P_t \end{bmatrix} + \epsilon_{t+1}$$

with

$$\epsilon_{t+1} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (0.17)^2 & -0.65(0.17)(0.0063) \\ -0.65(0.17)(0.0063) & (0.0063)^2 \end{bmatrix} \right)$$

and ϵ_{t+1} I.I.D.

(a) Discuss why the joint dynamics for log stock returns and the dividend yield imply mean-reversion for stock returns.

(b) How would mean-reversion be reflected in the optimal *static* allocation to equity of investors with different investment horizons.

(c) i. How would you expect the optimal *dynamic* allocation to equity to react to changes in the dividend yield. (Motivate your answer.)

   ii. Which is going the give the investor higher expected utility? The optimal static allocation or the optimal dynamic allocation? (Motivate your answer.)

5. Let the representatives investor's utility function be given by

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

The time discount factor of the investor is θ. Annual log consumption growth is conditionally normally distributed as

$$\Delta c_{t+1} = \mu_t + \sigma_c \epsilon_{t+1},$$

where μ_t is trend consumption growth at time t, σ_c is the standard deviation of consumption growth and $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$.

[1] A put option is a right to sell a share for a particular price.

(a) Show that the one year log risk-free rate is given by:

$$r_t^f = -\ln\theta + \gamma\mu_t - \frac{1}{2}\gamma^2\sigma_c^2$$

(b) Give an economic reason why:

i. the interest increases when θ increases.

ii. the interest increases when μ_t increases.

iii. the interest decreases when σ_c^2 increases.

We'll now be more specific about the process for log consumption growth. Trend consumption growth μ_t can take on two values depending on the state of the business cycle. State 1 is a boom state and state 2 is a recession state. The probability of going from a boom to a recession is given by p and the probability of going from a recession to a boom is q . Assume the following parameters:

$$\mu_t = \begin{cases} 0.03 & s_t = 1 \\ -0.02 & s_t = 2 \end{cases}$$
$$\sigma_c = 0.02$$
$$\theta = 0.99$$

(c) i. Find the one year risk-free rate in each of the two business cycle states for $\gamma = 1, \gamma = 75, \gamma = 125$

ii. Use your findings to sketch how the risk-free rate varies with the risk aversion parameter γ in each state of the economy.

(d) Assume $p = 0.25$ and $q = 0.5$ and use the law of iterated expectations to find the annualized two year log risk-free rate $r^{(2)}$ in both states of the economy for $\gamma = 1$.

According to the C-CAPM, the excess return to the market portfolio (the equity premium) should be given by:

$$EP = \gamma \text{cov}_t\left(r_{t+1}, \Delta c_{t+1}\right)$$

Assume that the correlation coefficient between the log return to the market portfolio and consumption growth is 0.1 and that the standard deviation of log returns to the market portfolio is 0.2.

(e) Find the γ necessary to generate the historical equity premium of 0.05. Relate your finding here and on part (c) to the equity premium and risk-free rate puzzle.

6. The Campbell-Shiller decomposition of expected stock returns is given by

$$r_{t+1} - E_t[r_{t+1}] = (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty}\rho^j r_{t+1+j}$$

Let log dividend growth be given by:

$$\Delta d_{t+1} = 0.02 + \epsilon_{t+1} + 0.7\epsilon_t \qquad \epsilon_t = \mathcal{N}(0, \sigma_2^2)$$

and ϵ_t I.I.D.

(a) i. Give a simplified expression for $\eta_{d,t+1} = (E_{t+1} - E_t)\sum_{j=0}^{\infty}\rho^j \Delta d_{t+1+j}$

ii. Find the impact of $\epsilon_{t+1} = 0.01$ on unexpected stock returns at time $t+1$.

(b) What do you expect the impact of ϵ_{t+1} to be on the log stock price at $t+1$? At $t+2$?

Important formulas

Vector derivatives

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x$$
$$\frac{\partial x'Ax}{\partial x} = (A + A')x$$

Inverses

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \qquad \text{(Special case)}$$

Expectations, variances

$$E_t[a + bx] = a + bE[x] \qquad \text{(Linearity)}$$
$$E_t[E_{t+1}[x]] = E_t[x] \qquad \text{(L.I.E.)}$$
$$\text{var}\,(x) = E[x^2] - E[x]^2$$
$$\text{var}\,(ax) = a^2\text{var}\,(x)$$
$$\text{cov}\,(x,y) = E[x\ y] - E[x]E[y]$$
$$\text{cov}\,(ax,y) = a\text{cov}\,(x,y) = \text{cov}\,(x,ay)$$

Expectation of a log-normal

$$E[e^x] = e^{\mu+\frac{1}{2}\sigma^2} \qquad \text{if} \qquad x \sim \mathcal{N}(\mu,\sigma^2)$$

MA(q) processes

If

$$x_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \cdots + \theta_q\epsilon_{t-q} \qquad \text{with } \epsilon_t \text{ I.I.D.}$$

Then

$$\text{var}\,(x_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = (\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q)\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \frac{\theta_j + \theta_1\theta_{j+1} + \theta_2\theta_{j+2} + \cdots + \theta_{q-j}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} \qquad \text{(auto-correlation)}$$

AR(1) process

If

$$x_t = \mu + \rho x_{t-1} + \epsilon_t \qquad \text{with } \epsilon_t \sim \mathcal{N}(0,\sigma),\ \epsilon_t\ I.I.D.$$

Then

$$\text{var}\,(x_t) = \frac{1}{1-\rho^2}\sigma^2 \qquad \text{(variance)}$$
$$\text{cov}\,(x_t, x_{t-j}) = \frac{\rho^j}{1-\rho^2}\sigma^2 \qquad \text{(auto-covariance)}$$
$$\phi_j = \rho^j \qquad \text{(auto-correlation)}$$

VR_k statistic

$$VR_k = \frac{1}{k}\frac{\text{var}\,(r_{t,t+k})}{\text{var}\,(r_{t+1})} = 1 + 2\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\phi_j,$$

where ϕ_j is the jth autocorrelation coefficient of returns.