Solution key, Investments 4.1 Exam, October 22, 2009

1. (a) See notes. Remember to set the derivative equal to 0.
   (b) $\begin{bmatrix}0.35 & -0.05\end{bmatrix}'$.
   (c) $E[R^p] = 1.036$, $\sigma(R^p) = 0.1612$. Compute either by the general formulas, or (more efficiently) by:

$$\begin{aligned} E[R^p] &= 1.01 + 0.35(0.08) - 0.05(0.04) \\ \sigma^2(R^p) &= (0.35)^2 0.25 + (-0.05)^2 0.25 + 2(0.35)(-0.05)0.15 \\ \sigma(R^p) &= \sqrt{\sigma^2(R^p)} \end{aligned}$$

   (d) $E[R^\star] = 1.0967$, $\sigma(R^\star) = 0.5357$. One way: Use the fact that the weights in tangency portfolio are proportional to the investors portfolio, so that

$$w^\star = 1/(0.3)\begin{bmatrix}0.35 & -0.05\end{bmatrix}' = \begin{bmatrix}7/6 & -1/6\end{bmatrix}'$$

   and plug these weights into the standard formula. Quicker way (maybe): Use the fact that the investor invests 70 % in risk free bonds and 30 % in the tangency portfolio, so that

$$\begin{aligned} R^p &= 0.7(R^f) + 0.3(R^\star) \qquad &\Rightarrow R^\star = 1/0.3(R^p - (0.7)R^f) \\ \sigma^2(R^p) &= (0.3)^2\sigma^2(R^\star) \qquad &\Rightarrow \sigma^2(R^\star) = (1/0.09)\sigma^2(R^p) \end{aligned}$$

   (e) i. On the MV-frontier, 30 % of the way from $(0, R^f)$ to the tangency portfolio.
      ii. At the tangency point.
      iii. At $(0, 1.01)$
      iv. On the RAF, highest one is below the tangency portfolio. (At least 95% of you placed the two stocks outside the RAF. This is not possible, because the whole RAF is spanned by the two stocks.)

2. (a) In up states: $m_{t+1} = 1/1.1$, in down states $m_{t+1} = 1/0.9$.
   (b) i. $\big((2/3)(1/1.1)\big)^2 = 0.3673$.
      ii. $\big((1/3)(1/0.9)\big)^2 = 0.1372$.
      iii. The first security is more expensive because it's more likely that we end up in the first state. If we look at the price weighted by the probability, the second security is more expensive. This is because marginal utility is higher in the "down"-"down" state. (At least half of you forgot to multiply the product of the SDF's with the path probabilities. Probably, this means that you didn't know exactly what you were doing, but since I forgot to multiply in the first solution I posted for the trial exam, I gave up to (3/4) of the full score if this applied to you.)
   (c) In the last period, starting from the top, the values are given by $0, 1, 1, 19$. In the second period, the value in the upstate is: $0.37 = (1/3)(1/0.9)(1)$. In the downstate it is 10. (Early exercise!) In the first period it is $3.93 = (2/3)(1/1.1)(0.37) + (1/3)(1/0.9)(10)$

3. (a) The investor invests the same amount into every security regardless of her wealth level. The question asks for "amount", not weight. To get the solution, you need to multiply with W_t. This is not plausible because we expect richer investors to hold more risky assets than poorer investors (your intuition/introspection should tell you so, but it is also what empirical evidence says.)
   (b) Replacing λ with aW_t, we have the same optimality condition as in section 1, since we already proved that the CAPM holds in this case, we are done. (This is the ideal solution that I hinted to in one of the review sessions. I think only one of you argued like this.) A more elaborate argument (that some made), would be that the optimality condition is the same as for mean-variance optimizers, so every investor picks portfolio's that are combinations of the risky-free assets and the tangency portfolio. This means the tangency portfolio will equal the market portfolio in equilibrium and the CAPM holds by the same derivation as in week 1.

4. (a) Because of the negative correlation between the two components of ϵ_{t+1}, positive return shocks will typically be associated with decreases in the dividend-yield. Through the top right element of the matrix of VAR coefficients, this will *on average* lead to lower returns at $t+2$. So the VAR dynamics implies a negative correlation between successive returns (mean-reversion).

   (b) The optimal static allocation to equity would be increasing with the investment horizon, because the expected return increases linearly with the investment horizon, while the variance increases slower than linearly.

   (c) i. When the dividend-yield goes up, the expected excess return to equity increases. Which makes equity more attractive. If the investor behaves optimally, he should allocate more to equity.[1]

   ii. The optimal dynamic allocation is better. An investor who maximizes dynamically could always choose the static asset mix. If he doesn't, it must be that the dynamic is better.

5. (a)

$$
\begin{aligned}
r_t^f &= -\ln E_t[m_{t+1}] = -\ln E_t[\theta e^{-\gamma \Delta c_{t+1}}] \\
&= -\ln\left(\theta \exp(-\gamma\mu_t + \frac{1}{2}\gamma^2\sigma_c^2)\right) \\
&= -\ln\theta + \gamma\mu_t - \frac{1}{2}\gamma^2\sigma_c^2
\end{aligned}
$$

   (b) Give an economic reason why:

   i. The representative investor is less impatient, lower compensation needed to postpone consumption, so r^f decreases. (The question asked you to explain why the interest rate increases with θ, this confused most of you, so I treated this as a bonus question. You got 1/3 by default and an extra (1/3) if you didn't get confused by the question and argued correctly.)

   ii. Consumption tomorrow higher, so marginal utility tomorrow lower. At unchanged r_t^f, investors would like to shift consumption from tomorrow to today. In equilibrium, the interest rate must increase to make it more expensive to do so.

   iii. When σ_c^2 increases, investors would like to save more to self-insure against bad consumption growth rates. (This is called precautionary saving.) By the same argument as above, this will drive down interest rates.

We'll now be more specific about the process for log consumption growth. Trend consumption growth μ_t can take on two values depending on the state of the business cycle. State 1 is a boom state and state 2 is a recession state. The probability of going from a boom to a recession is given by p and the probability of going from a recession to a boom is q . Assume the following parameters:

$$
\mu_t = \begin{cases} 0.03 & s_t = 1 \\ -0.02 & s_t = 2 \end{cases}
$$

$$
\sigma_c = 0.02
$$

   (c) i.

For boom state

$$
r_t^f = \begin{cases} 0.01 + 0.03 - \frac{1}{2}(0.02)^2 = 0.0398 & ,\gamma = 1 \\ 0.01 + 75(0.03) - \frac{1}{2}(75)^2(0.02)^2 = 1.135 & ,\gamma = 75 \\ 0.01 + 125(0.03) - \frac{1}{2}(125)^2(0.02)^2 = 0.635 & ,\gamma = 125 \end{cases}
$$

For recession state

$$
r_t^f = \begin{cases} 0.01 - 0.02 - \frac{1}{2}(0.02)^2 = -0.0102 & ,\gamma = 1 \\ 0.01 - 75(0.02) - \frac{1}{2}(75)^2(0.02)^2 = -2.615 & ,\gamma = 75 \\ 0.01 - 125(0.02) - \frac{1}{2}(125)^2(0.02)^2 = -5.625 & ,\gamma = 125 \end{cases}
$$

[1] Such an answer would get full points, but a truly complete argument would mention that the conditional variance of equity returns does not change with the dividend-yield. One can see this by staring at the equations, but I do not expect you to be experienced enough to see it.

For the boom state, the interest rate increases up to $\gamma = 75$ and then caves in. For the recession state the graph would be decreasing everywhere, but at an increasing rate.

ii.

iii. This is a tough one. Remember that the growth trend for $t+1$ is given by the current state (known), the uncertainty is about the growth trend for $t+2$, which depends on the state at $t+1$

$$\bar{m}_1 = 0.99e^{-0.03+0.5(0.02)^2} = 0.96 \quad (1)$$

$$\bar{m}_2 = 0.99e^{0.02+0.5(0.02)^2} = 1.01 \quad (2)$$

$$(3)$$

In state 1:

$$r^{(2)} = -\frac{1}{2}E_t[m_{t+1}m_{t+2}] = -\frac{1}{2}\ln\left((1-p)(\bar{m}_1)^2 + p(\bar{m}_1)(\bar{m}_2)\right) = 0.0335$$

In state 2:

$$r^{(2)} = -\frac{1}{2}E_t[m_{t+1}m_{t+2}] = -\frac{1}{2}\ln\left(q(\bar{m}_2)(\bar{m}_1) + (1-q)(\bar{m}_2)^2\right) = 0.0022$$

(d)

$$\begin{aligned} \text{cov}_t\left(r^m_{t+1}, \Delta c_{t+1}\right) &= (0.2)(0.1)(0.02) = 0.0004 \\ \Rightarrow\ \gamma &= (0.05/0.0004) = 125. \end{aligned}$$

For these parameters we need a risk aversion parameter of 125 to match the empirical equity premium of 5%. Such a parameter is economically implausible (it's extremely risk averse) and it would imply that the continuously compounded risk-free rate would move from 63.5 % in booms to -562.5 % when the economy goes from booms to recessions. Both numbers are completely crazy. (You can get nominal interest rates of 63.5 % during a hyper inflation, but never a real interest rate.) [2]

6. (a) i.

$$\eta_{d,t+1} = \epsilon_{t+1} + \rho 0.7\epsilon_{t+1}$$

ii.

$$\eta_{d,t+1} = (1+0.7\rho)0.01$$

(It would be nicer if I had said e.g. $\rho = 0.95$ so that you could really compute a number here.)

(b) At $t+1$ the price jumps to account for the higher dividends at all future horizons. (Notice Δd_{t+1} is a dividend *growth rate*; a shock to a growth rate has a permanent effect on the level of the variable.) At $t+2$, the shock is already incorporated in the stock price, so we would not expect a further jump.

[2] I see from your answers that there is some confusion about what this number really means: The gross risk-free rate would go from $R^f = 1.88$ to $R^f = 0.0036$. The way it is typically stated on your bank statement it would go from 88% to -99.74 %! If you weren't properly indoctrinated in primary school, I guess would be tempting to think that 5.625 means 5.625 %, but 5.625% = 5.625/100 = 0.05625, the percentage sign means take the number and divide it by 100.