Investments 4.1
Course code: 60412040

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Date: October 22, 2008
Time: 15:15 – 18:00
(2 hours, 45 minutes)

Parts:	The exam contains 20 parts. Some parts are divided into subparts. The subparts are numbered by (i), (ii), etc.
Grading:	Each part will give 3.5 points of the total 70.
Results:	Results will be known as soon as possible. At the latest Tuesday November 4.
Inspection:	You can inspect your marked exam papers Thursday, November 6, 15:30. The room will be announced.
Remark:	Provide complete answers, including computations where appropriate. On verbal questions: always provide motivation/explanation of your answer in terms of the economic mechanism at play. A short "yes" or "no" will never do as an answer. But be concise in your answers, otherwise you'll loose to much time writing it down.

Scan for the easiest questions and solve them first!

Useful formulas

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x \tag{1}$$

$$\frac{\partial x'Ax}{\partial x} = (A + A')x \tag{2}$$

$$\text{var}(x) = E[x^2] - E[x]^2 \tag{3}$$

$$\text{cov}(x, y) = E[x \cdot y] - E[x]E[y] \tag{4}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{5}$$

If $x \sim \mathcal{N}(\mu, \sigma^2)$, then

$$E[e^x] = e^{\mu + \frac{1}{2}\sigma^2} \tag{6}$$

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between to stocks and risk free bonds. The returns to the two stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. The investment opportunity set faced by investor is characterized by

$$R_f = 0.05 \qquad \mu = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix} \qquad \Omega = \begin{bmatrix} 0.04 & -0.01 \\ -0.01 & 0.04 \end{bmatrix}$$

The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$w'\mu + (1 - w'\iota)R_f - \frac{\lambda}{2} w'\Omega w$$

(a) Compute the optimal w for an investor with $\lambda = 2$.

$$w = \begin{bmatrix} 4/6 & 1/6 \end{bmatrix} \qquad \text{or} \qquad w = \begin{bmatrix} 0.6667 & 0.1667 \end{bmatrix}$$

(b) Compute the expected return and variance of the investor's portfolio.

$$E[R_p] = \quad 0.0833 \qquad \text{var}(R_p) = 0.0167$$

(c) The second stock has an expected return which is equal to the risk-free rate. Explain why it can make sense to hold such a stock in your portfolio.

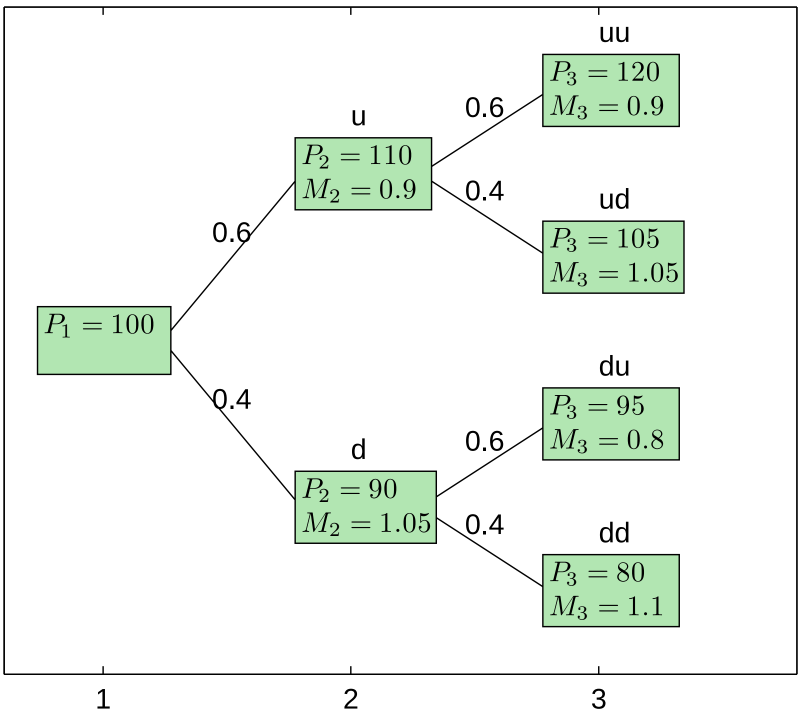
Because of its negative covariance with the first stock, including it in the portfolio reduces the total variance of the portfolio.

(d) Assume that the CAPM is true and that the value weighted supply of the two stocks are equal. (Value weighted supply = (Number of outstanding shares)×(current price per share)).

i. How would prices have to move in order for market equilibrium to obtain.
To increase relative demand for the second stock, its relative price needs to fall.

ii. How would this affect expected returns on the two stocks?
Relative to the expected return on the first stock, the expected return on the second stock will rise.

2. Consider the following simple economy. The probability of going up is equal to 60% always. P_t is the price of a stock in period t and M_{t+1} is the value of the stochastic discount factor between times t and $t+1$ depending on the state of the economy.



(a) Find the interest rate at time 1 and the interest rates in each of the nodes at time 2. Using $R_t^f = 1/E_t[M_{t+1}] - 1$, we find that $R_1^f = 0.0417$ $R_2^f|_u = 0.0417$ $R_2^f|_u = 0.08696$

(b) ST_t is the value at time t of a security that at time $t = 3$ pays out $|P_3 - 100|$.[1] Compute the value of the security at the nodes at $t = 2$ and $t = 1$. $ST_2|_u = 0.6 \cdot 0.9 \cdot 20 + 0.4 \cdot 1.05 \cdot 5 = 12.9$ the same kind of calculation gives $ST_2|_d = 11.2$ and $ST_1 = 11.67$

3. Let M_{t+1} denote the stochastic discount factor between time t and $t+1$. The Fundamental Asset Pricing Equation states that the return on any asset should be related to the stochastic discount factor M_{t+1}

$$1 = E_t[M_{t+1}(1 + R_{t+1})]$$

i. Use this equation to show that

$$E_t[R_{t+1}] = R_f - \frac{\text{cov}\,(M_{t+1}, R_{t+1})}{E_t[M_{t+1}]},$$

[1] It would e.g. pay out 0 if $P_3 = 100$ and 10 if $P_3 = 90$ or $P_3 = 110$. Such a payout structure is called a straddle.

where R_f is the risk free interest rate.

ii. Explain why expected returns depend negatively on the covariance between returns and the stochastic discount factor.

See slides and sample exam. Essentially, this is the same exercise.

4. Let log consumption growth, Δc_{t+1} be given by $\Delta c_{t+1} \sim \mathcal{N}(0.02, 0.01)$ and the stochastic discount factor be given by $M_{t+1} = 0.99e^{-\gamma \Delta c_{t+1}}$, where γ is the coefficient of relative risk aversion.

   (a) i. Find an expression for the continuously compounded risk-free rate (the log interest rate) using the relations $1 + R_f = 1/E_t[M_{t+1}]$ and $e^{r_f} = 1 + R_f$.

$$1 + R_f = \frac{1}{E_t[0.99]e^{-\gamma \Delta c_{t+1}}} = \frac{1}{E_t[0.99e^{-\gamma 0.02 + 0.5\gamma^2 0.01}]} = \frac{1}{0.99}e^{\gamma 0.02 - 0.5\gamma^2 0.01}$$
$$\Rightarrow r_f = -\log 0.99 + 0.02\gamma - 0.005\gamma^2$$

   ii. Compute the continuously compounded interest rate for $\gamma = 1$, $\gamma = 5$, and $\gamma = 25$.

   Plugging values into the formula above we find: $R_f = [0.0251\ \ -0.0149\ -2.6149]$, or $R_f = [0.0300\ 0.1088\ 0.4788]$ if you read 0.01 as the standard deviation. (Full points for both answers, as well as approximations of the kind $R_f = 0.01 + \gamma\mu_c$.)

   (b) The C-CAPM implies that the excess return of equity over risk free bonds by given by $\gamma \text{cov}\,(\Delta c_{t+i}, r_e)$, where r_e is the log return to equity. Assume the correlation coefficient between log equity returns and log consumption growth is 0.1 and that the standard deviation of equity returns is 0.2. Find the expected excess return for $\gamma = 1$, $\gamma = 5$ and $\gamma = 25$.
   Given the numbers above, the covariance is

$$\text{cov}\,(\Delta c_{t+i}, r_e) = 0.1 \cdot 0.2 \cdot 0.1 = 0.002$$

   multiplying with γ gives $EP = [0.0020\ 0.0100\ 0.0500]$ or $EP = [0.0002\ 0.0010\ 0.0050]$, depending on whether you are using $\sigma_c = 0.1$ or $\sigma_c = 0.01$

   (c) Relate your findings above to the equity premium puzzle and the risk-free rate puzzle.
   The equity premium puzzle says we need implausible levels of risk aversion to explain the historical equity premium. You see in point 2 that the EP is very low for low levels of γ. The risk free rate puzzle says that risk aversion has to be low to generate plausible interest rates.

5. In your case 3, we assumed that stock returns and the dividend yield were related through the VAR:

$$\begin{bmatrix} r_{t+1} \\ (D/P)_{t+1} \end{bmatrix} = \begin{bmatrix} -0.05 \\ 0.0033 \end{bmatrix} + \begin{bmatrix} 0.047361 & 3.5058 \\ 0.002202 & 0.875229 \end{bmatrix} \begin{bmatrix} r_t \\ (D/P)_t \end{bmatrix} + \epsilon_{t+1}$$

Assume the VAR describes the true relation between stock returns and the dividend yield. If your equilibrium model for equity would implies that returns should be IID (identically and independently distributed). What would this relationship between returns and the dividend yield tell you about market efficiency. The statistical relationship contradicts the equilibrium model, so either markets are inefficient or your model is wrong.

6. The variance ratio statistic for horizon k, VR_k, is computed as.

$$\text{VR}_k = \frac{1}{k}\frac{\text{var}\left(r_{t,t+k}\right)}{\text{var}\left(r_{t+1}\right)},$$

where $r_{t,t+k}$ is the log return to the asset between time t and k (i.e. $r_{t,t+k} = \sum_{j=1}^{k} r_{t+j}$). Let returns be generated by

$$r_{t+1} = 0.05 + \epsilon_t - 0.05\epsilon_{t-1}, \qquad \epsilon_t \sim \mathcal{N}(0, 0.2^2)$$

   (a) Compute the variance ratio statistics VR_2 and VR_3.
   There are many ways. Here's one:

$$\begin{aligned} \text{var}\left(r_{t+1}\right) &= \text{var}\left(\epsilon_t - 0.05\epsilon_{t-1}\right) = (1 + 0.05^2)0.2^2 \\ \text{cov}\left(r_{t,t+1}, r_{t+2}\right) &= -0.05 \times 0.2^2 \\ \text{cov}\left(r_{t+1}, r_{t+3}\right) &= 0 \end{aligned}$$

$$\begin{aligned} VR_2 &= \frac{2\text{var}\left(r_t\right) + 2\text{cov}\left(r_t, r_{t+1}\right)}{2\text{var}\left(r_t\right)} = 1 - \frac{0.05}{1 + 0.05^2} \approx 0.95 \\ VR_3 &= \frac{3\text{var}\left(r_t\right) + 4\text{cov}\left(r_t, r_{t+1}\right)}{3\text{var}\left(r_t\right)} = 1 - \frac{4}{3}\frac{0.05}{1 + 0.05^2} \approx 0.96 \end{aligned}$$

   (b) Empirical researchers tend to find that the Variance ratio statistic is higher than one for horizons shorter than 1 year, but ~~higher~~ lower than one for horizons larger than 1 year.[2]

      i. What does this finding tell us about the riskiness of equity in the long run relative to the short run.
      Equity is riskier in the short run.

      ii. How should an investor with CRRA utility (power utility), change her allocation to equity as she grows older if she trusts the empirical evidence?
      Reduce the allocation as she grows older. (Investment horizon gets shorter.) This is one of the key lessons from your case 2!

      iii. Compare your recommendation to that of Samuelson. Samuelson proves that the allocation should be independent of investment horizon for IID shocks. A VR statistic different from 1 indicates that returns are not IID, so Samuelson's results does not apply.

[2]This is obviously a misstatement. We learned in class that the VR is smaller than one at long horizons. I accepted a wide range of answers, as long as they were consistent on points (i) and (ii). If your score was lower on this point than the average of the other points. I did not count this question in your total grade. (I.e. your score on the other points were raised to account for one less part.)

(c) Describe a bootstrapping procedure that would let you compute significance levels for VR_k under the null that returns are independent, but with an unknown distribution.

Draw a large number of random samples from the return observations (with replacement) of the same length as the original series. By construction these are going to be to satisfy the null. Compute VR_k statistics for each sample, sort them, etc...

7. The Campbell-Shiller decomposition of unexpected returns is given by:

$$r_{t+1} - E_t[r_{t+1}] = (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j r_{t+1+j}$$

(a) What fraction of return variance would be explained by dividend growth rate news if:

i. Dividends follow a random walk with a drift. Only the fraction that comes from time $t+1$ dividend growth $(\Delta d_{t+1} - E[\Delta d_{t+1}])$. I also gave full points to 0 %, because it shows you got the idea of the question.

ii. Returns are I.I.D. (identically and independently distributed). 100 %. ($(E_t - E_{t-1})r_{t+j} = 0$, for all t and j.)

(b) Assume there is an unexpected structural break at time T in the required return to equity. Before time T, expected returns are given by $E_t[r_{t+j}] = 0.08$, for all $j > 1$. at T the expected return to equity decreases to 0.02. Investors expect the structural break to be permanent, so that $E_T[r_{T+j}] = 0.02$ for all $j > 1$. Expected dividend growth is unaffected by the structural break. Compute the unexpected return to equity at time T: $(r_T - E_{T-1}[r_T])$ given a coefficient $\rho = 0.99$.

$$\begin{aligned} -(E_T - E_{T-1})\sum_{j=1}^{\infty} \rho^j r_{T+j} &= -(E_T - E_{T-1})\sum_{j=1}^{\infty} 0.99^j (0.02 - 0.08) \\ &= -\frac{0.99}{1-0.99} \cdot (-0.06) = 596\%. \end{aligned}$$

(The economics behind this result is the following: Since there has been no change in expected dividends, the same dividend stream is now discounted at a much lower rate. This leads to a big unexpected jump in the price, corresponding to a huge surprise return at time T. You saw a similar effect in your second case.)

(c) Assume that at time t the price of equity drops by 50 % but there has been no news about future dividend growth. How does this influence your expectation about future returns according to the Campbell-Shiller decomposition? Can you tell how expected returns at different horizons are affected?

For sure they have to increase: You can now buy the same dividend-stream at half the price. From the accounting identity alone you cannot tell anything about how expected returns at different horizons are affected. (I didn't require the last point. Also your economic intuition would probably lead you to expect short term expected returns to be more affected than long term expected returns.)

8. An investor's value function at time t is given by

$$J_t(W_t) = \max_{C_t,\omega_t} \log C_t + \theta E_t[J_{t+1}(W_{t+1})]$$

The time $t+1$ value function is given by

$$J_{t+1}(W_{t+1}) = \frac{1}{1-\theta}\log W_{t+1} + k,$$

where k is a constant. The investor earns the stochastic return R_{t+1} on the fraction ω_t he invests in stocks and the risk-free rate R_f on what he invests in bonds. His wealth evolves according to $W_{t+1} = (W_t - C_t)(1 + (1-\omega_t)R_f + \omega_t R_{t+1})$.

(a) Find the optimal time t consumption level as a function of wealth.

Using the F.O.C for consumption at time t:

$$0 = \frac{1}{C_t} + \theta E_t[\frac{1}{1-\theta}\frac{1}{W_{t+1}}(-(1+(1-\omega_t)R_f + \omega_t R_{t+1}))]$$
$$= \frac{1}{C_t} - \frac{\theta}{1-\theta}\frac{1}{W_t - C_t}$$

Solving for C_t gives:

$$C_t = (1-\theta)W_t$$

(b) Show that the optimal allocation to stocks $\omega_t^\star$, is independent of the wealth level.

The F.O.C with respect to ω_t is $0 = \frac{\partial J_t(W_t)}{\partial \omega_t}$ or:

$$0 = \theta E_t\left[\frac{1}{1-\theta}\frac{1}{W_{t+1}}(R_e - R_f)\right]$$

Dividing by all terms that are constant or known at time t on both sides of the equation gives:

$$0 = E_t\left[\frac{R_e - R_f}{1+(1-\omega_t)R_f + \omega_t R_{t+1}}\right]$$

The current wealth level does not appear in this expectation, so the ω_t that solves the equation will not depend on W_t either.

(c) Substitute for the optimal consumption and $\omega_t^\star$ in the time t value function. Take $\omega_t^\star$ as a given in this exercise. Solve for $J_t(W_t)$ as a function of W_t. (Feel free to group all terms that are independent of wealth into a single variable).

Using $W_t - C_t = \theta W_t$ and $\log(x\, y) = \log x + \log y$:

$$J_t(W_t) = \log((1-\theta)W_t) + \theta E_t[\frac{1}{1-\theta}\log W_{t+1} + k]$$
$$= \log(1-\theta) + \log W_t + \theta E_t\left[\frac{1}{1-\theta}\left(\log\theta + \log W_t + \log(1+(1-\omega_t^\star)R_f + \omega_t^\star R_{t+1})\right) + k\right]$$
$$= \left(1 + \frac{\theta}{1-\theta}\right)\log W_t + k_t$$
$$= \frac{1}{1-\theta}\log W_t + k_t$$