Exam Applied Stochastic Modeling
19 December 2022, 8:30-11:15 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times$ $^{9}/_{45} + 1$.

Exercise 1. Two classes of customers arrive to an infinite-server station, which is open during $[0, 20]$. During the interval $[0, 20]$ customers of class 1 arrive according to a Poisson process with rate 5, and have an exponentially distributed service time with rate 1. Also, during the interval $[0, 20]$, customers of class 2 arrive according to a Poisson process with rate 10, and have an exponentially distributed service time with rate 2.
a. [2 pt.] What is the distribution of the total number of arrivals during $[0, 20]$ with corresponding parameter?
b. [4 pt.] Determine the expected number of customers of type i, for $i = 1, 2$, present at time $\tau \in [0, 20]$ ($m_i(\tau)$). What is the distribution of the total number of customers present at time 3 with corresponding parameter?
c. [3 pt.] Make a sketch of $m_i(\tau)$ for $\tau \in [0, 20]$ and $i = 1, 2$ in one figure. Intuitively explain whether $m_1(\tau)$ or $m_2(\tau)$ (for some fixed τ) is larger.

Exercise 2. Consider an M/G/1 queue with arrival rate 3/16. With probability $p \in (0, 1)$ the service time is exponentially distributed with rate $p/2$, and with probability $1 - p$ the service time is exponentially distributed with rate $(1 - p)/2$. Customers are served on a FCFS basis.
a. [4 pt.] Verify that the expected waiting time is

$$\mathbb{E}W_Q = \frac{3}{p(1-p)}.$$

b. [2 pt.] Make a sketch of the expected waiting time $\mathbb{E}W_Q$ as a function of p for $p \in [1/2, 1)$ and explain its behavior.
c. [4 pt.] To improve fairness, the service discipline is changed to SJF (Shortest Job First). We are interested in the expected waiting time for a customer of size x, i.e. $\mathbb{E}[W_Q(SJF) \mid S = x]$, for very small and very large x. Determine $\mathbb{E}[W_Q(SJF) \mid S = 0]$ and $\mathbb{E}[W_Q(SJF) \mid S = \infty]$ and compare the results with part a.

Exercise 3. Customers arrive to a single-server facility according to a Poisson process with rate 3. An arriving customer finding an empty system is always directly taken into service. If an arriving customer finds one other customer in the system upon arrival, the customer joins the system with probability 2/3 and leaves otherwise. An arriving customer always leaves the system when finding two or more customers present. Service times are exponentially distributed with rate 2.
a. [4 pt.] Draw the state diagram with the transition rates for the number of customers in the system and derive its stationary distribution.
b. [3 pt.] Give the distribution of the number of customers that an arriving customer joining the system sees upon arrival. What is the probability that a customer waits at least t time units and then receives service?
c. [3 pt.] Make a sketch of the number of customers in the system over time and give regeneration epochs. Let BP denote the busy period, that is, the time between the moment that a customer arrives to an empty system until the first moment the system is empty again. It can be derived that $\mathbb{E}BP = 1$. Use this in combination with renewal theory to derive the long-run fraction of time the server is idle.

Exercise 4. Consider a small production facility with production in two separate stages in series, both acting as single-server queues. Station 1 has unlimited waiting capacity, whereas station 2 has a total capacity of K (including the product in process). Orders arrive at station 1 according to a Poisson process with rate 4. After processing at station 1, products attempt to enter station 2. If all K storage spaces at station 2 are occupied, then the product is removed from the system. Processing times are exponentially distributed with rate μ_i at station i, for $i = 1, 2$.
a. [4 pt.] Draw the state diagram with corresponding transition rates for the two-dimensional Markov process of the number of products at both stations. Give the balance equations for states (n_1, K) with $n_1 > 0$, where n_1 denotes the number of products at station 1.
b. [4 pt.] Assume that $K = \infty$. Let $\mathbb{E}W_Q(i)$ be the expected waiting time at station i. Verify that the ratio of the waiting times $\mathbb{E}W_Q(1)/\mathbb{E}W_Q(2)$ equals $\mu_2(\mu_2 - 4)/(\mu_1(\mu_1 - 4))$ for $\mu_1, \mu_2 > 4$. Explain what happens with the ratio when $\mu_1 \to 4$.

Exercise 5. Consider a continuous-review continuous-product deterministic inventory model with holding costs, order costs and zero lead time (the EOQ model). Demand arrives at rate 5 per time unit. The holding costs are 1 per item per time unit. The total order costs depend on the order size, with the aim to avoid large orders at once. Specifically, when the order is Q, then the order costs are $10 + aQ + bQ^2$, where we choose $a = b = 3/10$ for now.
a. [4 pt.] Show that the total average costs per time unit for order size Q are

$$C(Q) = \frac{50}{Q} + 1.5 + 2Q.$$

b. [4 pt.] Determine the optimal order size Q^*. Management proposes to increase a in order to decrease the order size. Determine the optimal order size for arbitrary a and explain how effective the management proposal is.