Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

19 December 2022, 8:30-11:15 hours

Exercise 1.

a. Let $N_i(t)$ be the number of visitors of class i, $i = 1, 2$, during $[0, t]$ and let $N(t) = N_1(t) + N_2(t)$. As customers arrive according to a Poisson process during $[0, 20]$, it follows that $N_1(20)$ and $N_2(20)$ follow Poisson distributions with rates $5 \times 20 = 100$ and $10 \times 20 = 200$, respectively. Since the sum of Poisson random variables is again Poisson, $N(20)$ follows a Poisson distribution with rate $100 + 200 = 300$.

b. For the expected number of customers of type i at time $\tau \in [0, 20]$ ($m_i(\tau)$) (with λ_i and μ_i denoting the arrival and service rates, respectively), we have

$$m_i(\tau) = \int_0^\tau \lambda_i e^{-\mu_i(\tau - t)} \mathrm{d}t = \int_0^\tau \lambda_i e^{-\mu_i t} \mathrm{d}t = \frac{\lambda_i}{\mu_i} \left(1 - e^{-\mu_i \tau}\right),$$

such that $m_1(\tau) = 5(1 - e^{-\tau})$ and $m_2(\tau) = 5(1 - e^{-2\tau})$.

The number of class i customers present at time τ follows a Poisson distribution with rate $m_i(\tau)$. Hence, the total number of customers present at time 3 also follows a Poisson distribution with rate $m_1(3) + m_2(3) = 5(1 - e^{-3}) + 5(1 - e^{-6})$.

c. See Figure 1 for a sketch of $m_i(\tau)$ for $i = 1, 2$ and $\tau \in [0, 20]$. Observe that $m_2(\tau) \geq m_1(\tau)$. The offered load of both classes is the same (and equals 5), whereas $\mu_2 > \mu_1$ implying that the number of customers of class 2 converges faster to its steady state (more specifically, events for class 2 happen twice as fast).

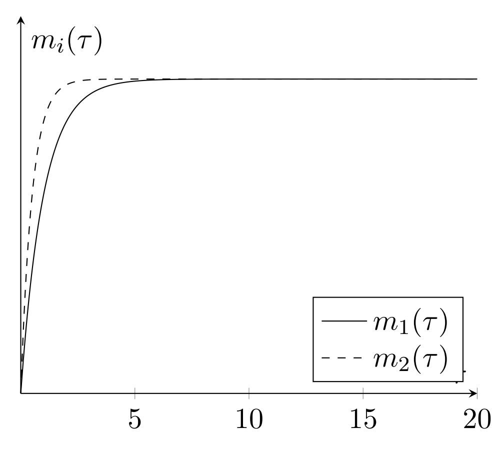

Figure 1: The evolution of $m_1(\tau)$ and $m_2(\tau)$ over time.

Exercise 2.

a. The first and second moment of the service time S are $\mathbb{E}S = p \times {}^2/_p + (1 - p) \times {}^2/_{(1-p)} = 4$

and

$$\mathbb{E}S^2 = p\frac{2}{(p/2)^2} + (1-p)\frac{2}{((1-p)/2)^2} = \frac{8}{p} + \frac{8}{1-p} = \frac{8}{p(1-p)}$$

The load is $\lambda\mathbb{E}S = 3/4$ (independent of p). The expected waiting time is thus given by

$$\mathbb{E}W_Q = \frac{\lambda\mathbb{E}S^2}{2(1-\lambda\mathbb{E}S)} = \frac{3/16 \times 8/(p(1-p))}{2(1-3/4)} = \frac{3}{p(1-p)}.$$

b. See Figure 2 for a sketch of the expected waiting time $\mathbb{E}W_Q$ as a function of $p \in [0.5, 1)$. Observe that as p increases, the variability in the service times increases, whereas the expectation (and also the load) remains the same. Hence, the waiting time increases with $p \in [0.5, 1)$. For $p \to 1$, the second moment of the service time explodes, and thereby also the expected waiting time.

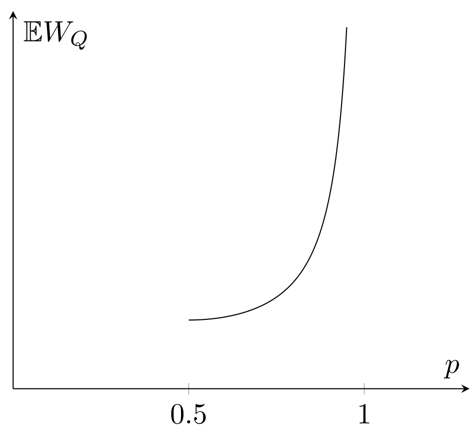

Figure 2: The expected waiting time $\mathbb{E}W_Q$ as a function of $p \in [0.5, 1)$.

c. For SJF, the expected waiting time for a customer of size x (with $f_S(t)$ the density of the service time) equals

$$\mathbb{E}[W_Q(SJF) \mid S = x] = \frac{\mathbb{E}R}{\left(1 - \lambda \int_0^x t f_S(t) \mathrm{d}t\right)^2},$$

where $\mathbb{E}R = \lambda\mathbb{E}S^2/2 = \frac{3}{4p(1-p)}$, see also part a. For $x = 0$ and $x = \infty$, we get

$$\begin{aligned}
\mathbb{E}[W_Q(SJF) \mid S = 0] &= \mathbb{E}R = \frac{3}{4p(1-p)} < \mathbb{E}W_Q(FCFS) \\
\mathbb{E}[W_Q(SJF) \mid S = \infty] &= \frac{\mathbb{E}R}{(1-\lambda\mathbb{E}S)^2} = \frac{12}{p(1-p)} > \mathbb{E}W_Q(FCFS),
\end{aligned}$$

with $\mathbb{E}W_Q(FCFS)$ corresponding to part a.

Exercise 3.

a. The transition diagram can be found in Figure 3. The balance equations are: $3\pi(0) = 2\pi(1)$ and $2\pi(1) = 2\pi(2)$. This yields $\pi(2) = \pi(1) = 3/2\ \pi(0)$. Using normalization, we find $\pi(0)$ from $\pi(0)\,[1 + 3/2 + 3/2] = 1$. Hence, $\pi(0) = 1/4$ and $\pi(1) = \pi(2) = 3/8$.

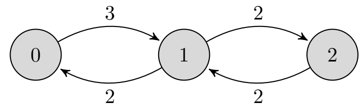

Figure 3: Transition diagram Exercise 3a.

b. For the distribution of the number of customers a joining customer sees upon arrival ($\alpha(x)$), we need 'beyond PASTA', i.e.,

$$\alpha(x) = \frac{\pi(x)\Lambda^{'}(x)}{\sum_{y=0}^{2} \pi(y)\Lambda^{'}(y)},$$

where $\Lambda^{'}(0) = 3$, $\Lambda^{'}(1) = 2$, and $\Lambda^{'}(2) = 0$. Hence, $\pi(0)\Lambda^{'}(0) = \pi(1)\Lambda^{'}(1) = {}^{3}\!/\!_{4}$. Combining the above yields $\alpha(0) = \alpha(1) = \frac{3/4}{3/4+3/4} = {}^{1}\!/\!_{2}$ and $\alpha(2) = 0$.

Finally, the probability that a joining customer waits at least t time units equals $\alpha(1)e^{-2t} = {}^{1}\!/\!_{2}e^{-2t}$.

c. Make a sketch of the number of customers in the system over time. As regeneration epochs, we take the moments that customers arrive to an empty system. A cycle then consists of a busy period (BP) and a consecutive idle period (I). Due to the renewal reward theorem, we obtain the long-run fraction of time the server is idle as

$$\text{Fraction server idle} = \frac{\mathbb{E}I}{\mathbb{E}BP + \mathbb{E}I} = \frac{1/3}{1 + 1/3} = \frac{1}{4}.$$

Exercise 4.

a. The transition diagram can be found in Figure 4. The balance equations, for $n_1 > 0$, are

$$(4 + \mu_1 + \mu_2)\pi(n_1, K) = 4\pi(n_1 - 1, K) + \mu_1\pi(n_1 + 1, K) + \mu_1\pi(n_1 + 1, K - 1).$$

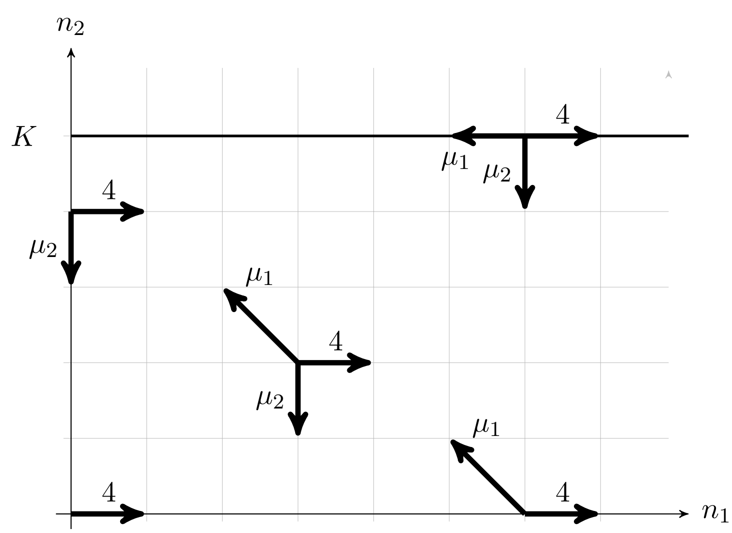

Figure 4: Transition diagram for Exercise 4a. Only outgoing transitions are shown.

b. Observe that, as $K \to \infty$, both stations behave as M/M/1 queues with arrival rate 4 (due to Burke's output theorem). Hence,

$$\mathbb{E}W_Q(i) = \frac{4/\mu_i}{\mu_i(1 - 4/\mu_i)} = \frac{4}{\mu_i(\mu_i - 4)}.$$

Thus, the ratio of waiting times yields

$$\frac{\mathbb{E}W_Q(1)}{\mathbb{E}W_Q(2)} = \frac{\frac{4}{\mu_1(\mu_1 - 4)}}{\frac{4}{\mu_2(\mu_2 - 4)}} = \frac{\mu_2(\mu_2 - 4)}{\mu_1(\mu_1 - 4)}.$$

When $\mu_1 \to 4$, this ratio explodes, as the first station tends to become unstable, such that the waiting time explodes (whereas the second station remains stable with a finite expected waiting time).

Exercise 5.

a. The inventory process may be considered as a regenerative process, with the order moments with an empty inventory as regeneration epochs (a sketch of the inventory process is convenient). The cycle length equals $T = Q/5$. The holdings costs per cycle are $1 \times Q \times T \times 1/2$. The order costs per cycle are clearly $10 + aQ + bQ^2$ (you may directly substitute $a = b = 3/10$). Hence, the renewal reward theorem provides the long-run average cost per time unit

$$C(Q) = \frac{10 + aQ + bQ^2}{Q/5} + \frac{1}{2}Q = \frac{50}{Q} + 5a + \left(\frac{1}{2} + 5b\right)Q,$$

which provides the desired result for $a = b = 3/10$.

b. Taking the derivative of $C(Q)$ with respect to Q gives

$$C'(Q) = -\frac{50}{Q^2} + 2.$$

Solving $C'(Q) = 0$ yields the optimal order size $Q^* = \sqrt{25} = 5$; note that we have a global minimum as $C''(Q) > 0$.

Observe that the derivative $C'(Q)$ remains the same for an arbitrary a, and thus $Q^* = 5$ remains optimal. Hence, the management proposal does not influence the order size.