Exam Applied Stochastic Modeling
20 December 2021, 8:30-11:15 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times$ $^9/_{32} + 1$.

Exercise 1. There is an open air event during the time interval $[0, 6]$. New visitors only arrive during the time interval $[0, 4]$. In particular, new visitors arrive according to a Poisson process with rate λ_1 during $[0, 2]$, whereas new visitors arrive according to a Poisson process with rate λ_2 during $(2, 4]$. We assume that visitors stay exactly for 2 hours.
a. [2 pt.] What is the expected number visitors arriving to the open air event during $[0, 4]$? And what is the corresponding distribution with corresponding parameter?
b. [3 pt.] Determine the expected number of visitors present at the event at time τ ($m(\tau)$) and verify that $m(\tau) = \lambda_1\tau$ for $\tau \in [0, 2]$, and $m(\tau) = \lambda_1(4 - \tau) + \lambda_2(\tau - 2)$ for $\tau \in (2, 4]$. Moreover, determine $m(\tau)$ for $\tau \in (4, 6]$.
c. [3 pt.] Suppose that the total expected number of visitors is 600. Using a special promotion, the organizers can influence whether visitors arrive during $[0, 2]$ or during $(2, 4]$. That is, they can determine λ_1 and λ_2 as long as the total expected number of arrivals during $[0, 4]$ is 600. For which values of λ_1, λ_2 is the peak in the number of visitors minimized (i..e., $\min \max m(\tau)$)? Make a sketch of $m(\tau)$ for $\tau \in [0, 6]$ and explain its behavior.

Exercise 2. Jobs arrive at a processing facility according to a Poisson process with arrival rate λ. Upon arrival, jobs are randomly assigned to one of the two separate servers, i.e., a job is assigned to server i, $i = 1, 2$, with probability 1/2 independent of any other job. The service times are exponentially distributed with rate 1.
a. [2 pt.] Determine the expected waiting time.

To avoid long waiting times, management decides to change the routing policy. Jobs smaller than t are sent to queue 1 and jobs of at least size t are sent to queue 2.
b. [3 pt.] We focus on queue 1. Argue that queue 1 can be considered as an M/G/1 queue, and determine the expected waiting time.
Hint: Without proof, you may use that $\int_a^b xe^{-x}\mathrm{d}x = (a + 1)e^{-a} - (b + 1)e^{-b}$, and $\int_a^b x^2e^{-x}\mathrm{d}x = (a^2 + 2a + 2)e^{-a} - (b^2 + 2b + 2)e^{-b}$.

c. [1 pt.] Let $t = 1$. Give the stability condition for queue 1 in terms of λ.

Exercise 3. The lifetime of a machine is uniformly distributed between 0 and b. The maintenance policy is as follows: (as good as) new machines are inspected after t time units. If the machine is running at inspection, there is a small update; the time required for this update can be neglected. If the machine is down, then there is a repair taking 2 time units. After an update or repair, the machine is as good as new. An inspection costs K and a repair costs 100. In addition, there is a cost of 10 for every time unit that the machine is down before the machine is inspected.
a. [4 pt.] Make a sketch of the state of the machine over time and give regeneration epochs. Use renewal theory to show that the long-run average costs are

$$\frac{1}{b+2}\left(\frac{Kb}{t} + 5t + 100\right).$$

b. [2 pt.] Determine the optimal inspection time t.

Exercise 4. Consider a single-server queue with a Poisson arrival process with rate 3. The service times are exponentially distributed. When there are at most three customers, the service rate is μ_1, whereas the service rate is μ_2 when there are four or more customers in the system.
a. [4 pt.] Draw the state diagram with the transition rates for the number of customers in the system and give the balance equations. Let $\mu_1 = 3$. Derive the stationary distribution of the number of customers in the system (in terms of μ_2).

After service, customers leave the system with probability 1/3, or join a second queue otherwise. This second queue has a single server, where service times follow an exponential distribution with rate 4. After service at the second queue, customers leave the system.
b. [2 pt.] Let $\mu_1 = \mu_2 > 3$. Give the stationary distribution of the joint number of customers in the system. Has the system a product-form solution when $\mu_1 \neq \mu_2$?
c. [2 pt.] Draw the state diagram with corresponding transition rates for the two-dimensional Markov process of the number of customers at both stations (for $\mu_1 \neq \mu_2$).

Exercise 5. To start a new company, an initial capital needs to be raised. The amount of capital required before the company is profitable is unknown, and follows a random variable D. For the initial capital S raised, you pay an interest rate r per euro. If more capital is required, then you pay interest rate $2.5r$ per euro over the amount you require on top of S.
a. [2 pt.] Formulate the expected cost as a function of the amount of initial capital raised.
b. [2 pt.] Suppose that D follows a normal distribution with mean 200 and standard deviation 80. A consultant advises $S = 200$. Use a marginal argument to argue whether this is too high, too low, or optimal.