Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

20 December 2021, 8:30-11:15 hours

Exercise 1.

a. Let $N(t)$ be the number of visitors during $[0,t]$. Then $\mathbb{E}N(4) = \int_0^2 \lambda_1 \mathrm{d}t + \int_2^4 \lambda_2 \mathrm{d}t = 2(\lambda_1+\lambda_2)$. Moreover, as visitors arrive according to a (time-dependent) Poisson process, it follows that $N(4)$ follows a Poisson distribution, with rate $2(\lambda_1+\lambda_2)$.

b. The number of visitors at time τ consists of the arrivals during $[(\tau-2)^+,\tau]$. Hence, for $\tau \in [0,2]$, we have

$$m(\tau) = \int_0^\tau \lambda_1 \mathrm{d}t = \lambda_1 \tau.$$

For $\tau \in (2,4]$, we have

$$m(\tau) = \int_{\tau-2}^2 \lambda_1 \mathrm{d}t + \int_2^\tau \lambda_2 \mathrm{d}t = \lambda_1(4-\tau) + \lambda_2(\tau-2).$$

Finally, for $\tau \in (4,6]$, we obtain

$$m(\tau) = \int_{\tau-2}^4 \lambda_2 \mathrm{d}t = \lambda_2(6-\tau).$$

c. Note that it is assumed that $2(\lambda_1+\lambda_2) = 600$, hence, $\lambda_1+\lambda_2 = 300$. Observe that the two candidate instants at which $m(\tau)$ attains a peak are the moments 2 and 4, i.e., $\max_{\tau\in[0,6]} m(\tau) = \max\{m(2), m(4)\} = \max\{2\lambda_1, 2\lambda_2\}$. This value is smallest if $m(2) = m(4)$, thus, for $\lambda_1 = \lambda_2$. This implies that $\lambda_1 = \lambda_2 = 150$. See Figure 1 for the corresponding sketch of $m(\tau)$ for $\tau \in [0,6]$.

Figuur 1: The evolution of $m(\tau)$ over time for $\lambda_1 = \lambda_2 = 150$.

Exercise 2.
a. Due to random splitting and thinning of a Poisson process, we have two identical M/M/1 queue with arrival rate $\lambda/2$. As $\mu = 1$, we have load $\rho = \lambda/2$. The expected waiting time is

$$\mathbb{E}W_Q = \frac{\lambda/2}{1 - \lambda/2} = \frac{\lambda}{2-\lambda}.$$

b. Again, due to thinning of a Poisson process, queue 1 has Poisson arrivals with rate $\lambda(1-e^{-t})$. The service times are truncated exponential (truncated at t), which thus follows a general distribution, hence, queue 1 behaves as an M/G/1. Now,

$$\begin{aligned}\mathbb{E}[S;\ \text{type 1}] &= \int_0^t x e^{-x}\mathrm{d}x &= 1-(t+1)e^{-t} \\ \mathbb{E}[S^2;\ \text{type 1}] &= \int_0^t x^2 e^{-x}\mathrm{d}x &= 2-(t^2+2t+2)e^{-t}\end{aligned}$$

Thus, the expected waiting time at queue 1 is

$$W_Q(1) = \frac{\lambda \mathbb{E}[S^2;\ \text{type 1}]}{2(1-\mathbb{E}[S;\ \text{type 1}])} = \frac{\lambda\left(2-(t^2+2t+2)e^{-t}\right)}{2\left(1-\lambda(1-(t+1)e^{-t})\right)}$$

c. For $t = 1$, note that the load of queue 1 is $\lambda\left(1-2e^{-1}\right)$. Hence, after some rewriting, it follows that queue 1 is stable if and only if $\lambda < \frac{e}{e-2}$.

Exercise 3.
a. Make a sketch. Regeneration epochs are, for instance, moments when the machine is (as good as) new; we use these regeneration epochs below [1]. Then, the expected cycle length is $\mathbb{E}T = t + 2 \times t/b = t(1+2/b)$. For the expected cost per cycle, we have

- Inspection cost: K
- Repair cost: $100 \times t/b$
- Downtime cost: $10\int_0^t (t-x)\frac{1}{b}\mathrm{d}x = t^2\ 5/b$

Using the renewal reward theorem, the long-run average costs per time unit $C(t)$ are

$$C(t) = \frac{K + 100t/b + 5t^2/b}{t(1+2/b)} = \frac{1}{b+2}\left(\frac{Kb}{t} + 5t + 100\right).$$

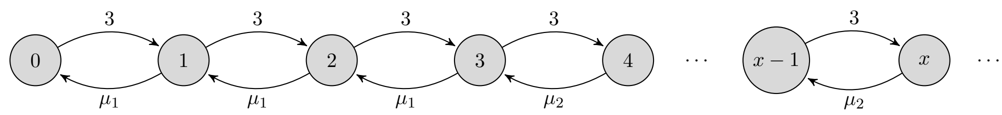
b. Taking the derivative of $C(t)$ with respect to t gives

$$C'(t) = \frac{1}{b+2}\left(5 - \frac{Kb}{t^2}\right).$$

Solving $C'(t) = 0$ yields the optimal inspection time $t^* = \sqrt{Kb/5}$; note that we have a global minimum as $C''(t) > 0$.

Exercise 4.
a. Let $X(t)$ be the number of customers at time t. The corresponding transition diagram can be found in Figure 2.

[1] Another option is to take the moments just after a repair, but this complicates the analysis.



Figuur 2: Transition diagram Exercise 4a.

The balance equations (for sets) are: $3\pi(x) = \mu_1\pi(x+1)$, for $x = 0, 1, 2$, and $3\pi(x) = \mu_2\pi(x+1)$, for $x = 3, 4, \ldots$. Now, let $\mu_1 = 3$. Then, $\pi(3) = \pi(2) = \pi(1) = \pi(0)$, and, for $x = 3, 4, \ldots$,

$$\pi(x) = \frac{3}{\mu_2}\pi(x-1) = \left(\frac{3}{\mu_2}\right)^{x-3}\pi(3) = \left(\frac{3}{\mu_2}\right)^{x-3}\pi(0).$$

Using normalization, we find $\pi(0)$ from

$$\pi(0)\left[3 + \sum_{x=3}^{\infty}\left(\frac{3}{\mu_2}\right)^{x-3}\right] = 1.$$

After some rewriting, we obtain

$$\pi(0) = \frac{\mu_2 - 3}{4\mu_2 - 9}.$$

b. The routing equations yield $\gamma_1 = 3$ and $\gamma_2 = \frac{2}{3}\gamma_1 = 2$. Thus, for $\mu_1 = \mu_2 > 3$, we have

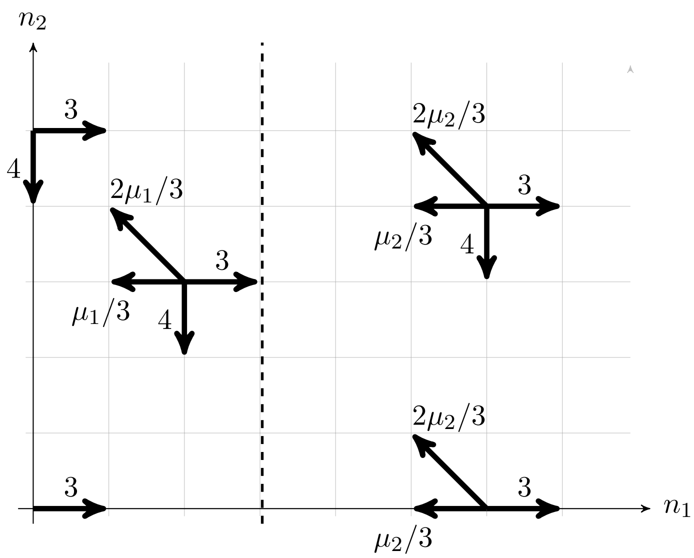
$$\begin{aligned}\pi(n_1, n_2) &= \left(1 - \frac{3}{\mu_1}\right)\left(\frac{3}{\mu_1}\right)^{n_1}\left(1 - \frac{2}{4}\right)\left(\frac{2}{4}\right)^{n_2} \\ &= \left(1 - \frac{3}{\mu_1}\right)\left(\frac{3}{\mu_1}\right)^{n_1}\frac{1}{2}\left(\frac{1}{2}\right)^{n_2}.\end{aligned}$$

For $\mu_1 \neq \mu_2$, $\pi(n_1, n_2)$ also has a product-form solution, as it is a generalized Jackson network.
c. The transition diagram can be found in Figure 3.

Exercise 5.
a. The expected cost $C(S)$ as a function of the amount of capital raised S is

$$C(S) = rS + 2.5r\mathbb{E}(D - S)^+,$$

where the first term is the cost for the initial capital raised and the second term corresponds to the expected additional capital.
b. Marginal argument: initially raise the Sth unit costs r. *Not* initially raising the Sth unit costs $2.5r\mathbb{P}(D \geq S)$. Note that, for $S = 200$, the latter equals $2.5r \times 0.5 = 1.25r > r$. Hence, raising the 200th unit costs less than not raising it, and $S = 200$ is too low.



Figuur 3: State diagram for Exercise 4c. Only outgoing transitions are shown.