Exam Applied Stochastic Modeling
8 February 2021, 18:45-21:30 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times$ $9/30 + 1$.

Exercise 1. In a specific region, there is a test location for getting tested for the coronavirus. Essentially, the test location can be considered as a queue with a single server. People with symptoms arrive to the test location according to a Poisson process with rate 14 per hour. The expected time for testing is 4 minutes and the standard deviation is 2 minutes.
a. [2 pt.] Determine the expected waiting time.

The government decides that some groups get priority for testing. Assume that a fraction $p \in (0, 1)$ of the people that get tested receive priority.
b. [3 pt.] Determine the expected waiting times of both groups as a function of p. Explain what happens with the waiting time of the priority group when $p \to 0$ and explain the actual value? And what happens when $p \to 1$?

Exercise 2. We are interested in the vaccination process of a first batch of vaccines of some pharmaceutical company. This batch arrives during the next 4 weeks, i.e., for $t \in [0, 4]$. Vaccines are assumed to arrive according to an inhomogeneous Poisson process with rate (in millions)

$$\lambda(t) = 2t - \frac{1}{2}t^2,$$

Vaccines are kept on stock until vaccination. Vaccination happens exactly one week after arrival of the vaccine. There are no vaccines on stock at time 0.
a. [3 pt.] What is the expected number of vaccines that arrive during $[0, 4]$? And what is the corresponding distribution? Give an approximation of the probability that less than $16/3 - 8/3\sqrt{3}$ vaccines arrive during these 4 weeks.
b. [2 pt.] Determine the expected number of vaccines on stock at time $\tau \in [0, 4]$. Distinguish the cases $\tau \in [0, 1]$, and $\tau \in [1, 4]$.
c. [3 pt.] Argue that the number of people that have been vaccinated during $[0, \tau]$, for $\tau \in [0, 5]$, follows a Poisson distribution and determine the corresponding parameter.

Exercise 3. A small company with N employees has two cottages that employees may rent (e.g. for holidays or business contacts). The time that an employee rents a cottage is exponentially distributed with rate 1. The time until an employee wants to rent a cottage again is exponentially distributed with rate 0.2, independent of past rentals. If both cottages are occupied, the employees are put on a waiting list and use the cottage on a FCFS basis.
a. [3 pt.] Model the availability of the cottages as a birth-and-death process, and draw the state diagram with corresponding transition rates.
b. [1 pt.] Can the system be interpreted as a closed Jackson network? If so, make a sketch of the network.

Exercise 4. Consider a queue (named Q1) with 2 servers. Customers arrive according to a Poisson process with rate 2. Service times are exponential with rate 2.
a. [3 pt.] Draw the state diagram with the transition rates, and derive the stationary distribution.

After being served at Q1, customers go to a second queue (named Q2) with probability $p \in [0, 1]$, and leave otherwise. The second queue (Q2) has a single server having exponential service times with rate 3. Also, new external customers arrive to Q2 according to a Poisson process with rate 1.
b. [3 pt.] Formulate the routing equations for this system. For which values of p is the system stable? Let $\pi(n_1, n_2)$ denote the stationary distribution of the joint number of customers. Give $\pi(0, n_2)$ and $\pi(n_1, n_2)$ for $n_1 \geq 1$, $n_2 \geq 0$, in terms of p.
c. [1 pt.] Let $p \in (0, 1)$. You have the option to move one server from Q1 to Q2. Would you take this option?

Exercise 5. Consider a continuous-review continuous-product deterministic inventory model with holding costs h per item per time unit, order costs K per order and zero lead time (the EOQ model). Demand arrives continuously at rate 1 per time unit.
a. [2 pt.] Give the optimal order size and corresponding average costs per time unit. Verify that there is a unique optimal order size.

There turns out to be variability in the received order. When the order size is Q, then the received order quantity is a random variable with expectation Q and standard deviation $\sigma \times Q$.
b. [4 pt.] Make a sketch of the inventory level over time, and define convenient regeneration epochs. Use renewal theory to derive that the average costs per time unit are

$$C(Q) = \frac{K}{Q} + \frac{1}{2}h(\sigma^2 + 1)Q.$$

Determine the optimal order quantity.