Exam Applied Stochastic Modeling
14 December 2020, 8:30-11:15 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points × 9/29 + 1.

Exercise 1. We are interested in the development of the number of Covid patients at the IC. Covid patients are assumed to arrive at the IC according to an inhomogeneous Poisson process. Over the interval $[0, 60]$, the number of arrivals increases and the arrival rate is estimated to be $\lambda(t) = e^{t/15}$, for $t \in [0, 60]$. At time 60 measures are taken. Suppose that the arrival rate of Covid patients instantaneously is $\lambda(t) = Ce^4$, for $t > 60$. For simplicity, Covid patients are assumed to be exactly 15 time units at the IC. At time 0, there were no Covid patients at the IC.
a. [2 pt.] What is the expected number of Covid patients arriving to the IC during $[0, 60]$? And what is the corresponding distribution with corresponding parameter?
b. [3 pt.] Determine the expected number of Covid patients at the IC at time τ ($m(\tau)$) and verify that $m(\tau) = 15(e^{\tau/15} - 1)$ for $\tau \in [0, 15]$ and $m(\tau) = 15e^{\tau/15}(1 - e^{-1})$ for $\tau \in (15, 60]$. Moreover, verify that $m(\tau) = 15(e^4 - e^{\tau/15}e^{-1}) + Ce^4(\tau - 60)$ for $\tau \in (60, 75]$.
c. [3 pt.] Let $C = 1$. Make a sketch of $m(\tau)$ for $\tau \in [0, 75]$ and explain what happens with the expected number of Covid patients at the IC during the period $[0, 75]$. Do the same for $C = 0$.

Exercise 2. Customers arrive to a small specialized shop according to a Poisson process with rate 0.9. The service time consists of two phases: taking the order and delivery of the order. The time for taking orders is exponentially distributed with rate 3/2, whereas the time for delivery is exponentially distributed with rate 3. There is a single server.
a. [2 pt.] Verify that the total expected service time is 1. Determine the expected waiting time.
b. [2 pt.] In the period before Christmas, the arrival rate is 10% higher. Determine the relative increase in expected waiting time and argue why this differs from 10%.

Due to Covid measures, there can now only be one customer in the shop. For simplicity, we assume that arriving customers that find a customer in the shop go elsewhere and are lost.
c. [3 pt.] Use renewal theory to determine the fraction of time that the server is working. How much smaller is this compared to the situation in part a? What is the fraction of lost sales?

Exercise 3. Consider a birth-and-death process with birth rate λ_x, $x = 0, 1, \dots$ and death rate μ_x, $x = 1, \dots$.
a. [2 pt.] Suppose that the birth and death rates are constant. To which queue does this birth-and-death process correspond? Answer the same question in case the birth rate is constant and $\mu_x = \min\{x, 4\}\mu$.
b. [4 pt.] Suppose that $\lambda_x = \lambda/(x+1)$ and the death rate is constant. Draw the transition diagram with the corresponding transition rates and derive the stationary distribution.

Exercise 4. Consider an open queueing network consisting of three queues, each having a single server. New customers arrive to queue 1 according to a Poisson process with rate 1. After being served at queue 1, customers go to queue 2 with probability p and to queue 3 with probability $1 - p$. After being served at queue 2, customers go to queue 1 with probability 3/4 and leave the network otherwise. After being served at queue 3, customers leave the network. The service times at queues 1, 2, and 3 are exponential with rate 2.
a. [3 pt.] Formulate the routing equations for this system. For which values of p is the system stable? Also give the stationary distribution of the joint number of customers in the system.

Exercise 5. A pharmaceutical company aims to produce a vaccine. The company will receive a single order for D vaccines from the government, where D is random (due to competition with other vaccines). The company has to decide beforehand how much to produce. For the vaccines that are sold to the government, the company receives p per vaccine. Vaccines that cannot be sold to the government have to be sold elsewhere, for which they receive 10% of the price, i.e., $0.1p$ per vaccine. There is a fixed cost for production and research and development K. The variable production cost is $0.2p$ per vaccine.
a. [2 pt.] Argue that the profit $P(S)$ when producing S vaccines is

$$P(S) = 0.8p\mathbb{E}[\min\{D, S\}] - 0.1p\mathbb{E}(S - D)^+ - K.$$

Give an expression for the optimal production size.
b. [3 pt.] Suppose that the demand from the government is uniformly distributed between 0 and 50 (in millions). Show that the expected number of vaccines that the company is short (i.e. the expected number of lost sales) is $\mathbb{E}(D - S)^+ = (50 - S)^2/100$, for $S \in [0, 50]$. What is the fraction of lost sales?