Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

14 December 2020, 8:30-11:15 hours

Exercise 1.

a. Let $N(t)$ be the number of arriving Covid patients to the IC during $[0, t]$. Then $\mathbb{E}N(60) = \int_0^{60} e^{t/15}\mathrm{d}t = 15(e^4-1)$. Moreover, as patients arrive according to a (time-dependent) Poisson process, it follows that $N(60)$ follows a Poisson distribution, with rate $15(e^4 - 1)$.

b. The number of Covid patients at the IC at time τ consists of the arrivals during $[(\tau - 15)^+, \tau]$. Hence, for $\tau \in [0, 15]$, we have

$$m(\tau) = \int_0^\tau e^{t/15}\mathrm{d}t = 15(e^{\tau/15} - 1).$$

For $\tau \in (15, 60]$, we have

$$m(\tau) = \int_{\tau-15}^\tau e^{t/15}\mathrm{d}t = 15e^{\tau/15}(1 - e^{-1}).$$

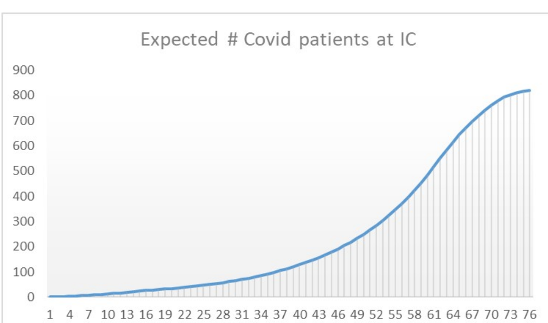
Finally, for $\tau \in (60, 75]$, we obtain

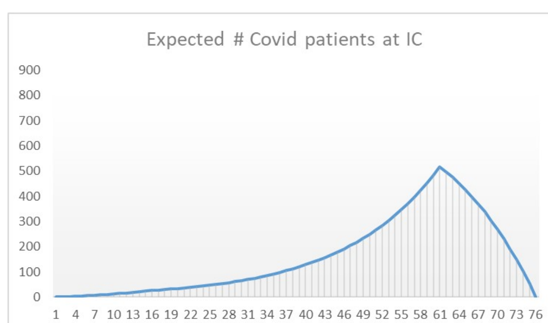
$$m(\tau) = \int_{\tau-15}^{60} e^{t/15}\mathrm{d}t + \int_{60}^\tau Ce^4\mathrm{d}t = 15(e^4 - e^{\tau/15}e^{-1}) + Ce^4(\tau - 60).$$

c. See Figure 1 for a sketch of $m(\tau)$ for $\tau \in [0, 75]$. Note that there is an exponential increase in $m(\tau)$ during $[0, 60]$ due to the exponential increase in the number of arrivals.

If $C = 1$, then the arrival rate remains constant at its peak. In that case, $m(\tau)$ keeps increasing, and converges to its equilibrium at $15Ce^4$. Note that this happens gradually, due to the delay in $m(t)$ compared to $\lambda(t)$ in view of the length of stay of 15 days.

If $C = 0$, there are no more arrivals. As the arrival rate is thus smaller than the arrival rate at time $60 - 45 = 15$, $m(\tau)$ decreases instanteneously. Now, $m(\tau)$ converges to 0.





Figuur 1: The evolution of $m(\tau)$ over time for $C = 1$ (left) and $C = 0$ (right).

Exercise 2.
a. Denote by S_1 and S_2 the time for taking orders and delivery of orders, respectively. Then, the expected service time $\mathbb{E}S$ equals $\mathbb{E}S = \mathbb{E}S_1 + \mathbb{E}S_2 = 2/3 + 1/3 = 1$.

For the waiting time, note that $\mathbb{V}\text{ar}S = \mathbb{V}\text{ar}S_1 + \mathbb{V}\text{ar}S_2 = 1/(3/2)^2 + 1/3^2 = 5/9$. Hence, $\mathbb{E}S^2 = \mathbb{V}\text{ar}S + (\mathbb{E}S)^2 = 14/9$ and for the residual service time R we obtain $\mathbb{E}R = 0.9\,\mathbb{E}S^2/2 = 7/10$. Now, the expected waiting time is

$$\mathbb{E}W_Q = \frac{\mathbb{E}R}{1-\rho} = \frac{7/10}{1 - 9/10} = 7.$$

b. The new arrival rate is $0.9 \times 1.1 = 0.99$. Thus, we now have $\mathbb{E}R = 77/100$ and the new waiting time equals

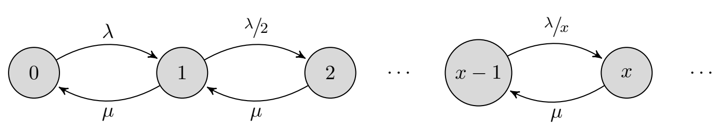
$$\mathbb{E}W_Q = \frac{77/100}{1 - 99/100} = 77.$$

The expected waiting time thus increases by 1100%. This is due to the impact of the load, leading to a (highly) non-linear increase. Specifically, just before Christmas the load of the system is very high leading to excessive waiting.
c. Let the moments that a customer leaves be the regeneration epochs. A regeneration cycle then consists of an interarrival time, S_1 and S_2, successively. Hence, the expected cycle length is $\mathbb{E}T = 10/9 + 2/3 + 1/3 = 19/9$. Impose a reward of 1 when the server is working. Using the renewal reward theorem, we obtain

$$\text{Fraction of time working} = \frac{\mathbb{E}[\text{working per cycle}]}{\mathbb{E}T} = \frac{1}{19/9} = \frac{9}{19}.$$

For part a, the fraction of time the server is working is $9/10$ (due to Little's law); hence, the fraction of time the server is working decreased with $9/10 - 9/19 = 81/190$. Due to PASTA, the fraction of lost sales is $9/19$.

Exercise 3.
a. The case of constant birth and death rates corresponds to the M/M/1 queue.
The case of a constant birth rate and death rate $\mu_x = \min\{x, 4\}\mu$ corresponds to the M/M/4 queue with service rate μ.
b. The transition diagram of this birth-and-death process can be found in Figure 2.



Figuur 2: Transition diagram Exercise 3b.

The stationary distribution follows from the balance equations (for sets): $\lambda/x\ \pi(x-1) = \mu\pi(x)$, for $x = 1, 2, \ldots$. Hence,

$$\pi(x) = \frac{\lambda}{\mu}\frac{1}{x}\pi(x-1) = \left(\frac{\lambda}{\mu}\right)^x \frac{1}{x!}\pi(0) \qquad x = 0, 1, \ldots.$$

Using normalization, we obtain that

$$\pi(0) = \left[\sum_{x=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^x \frac{1}{x!}\right]^{-1} = e^{-\lambda/\mu}.$$

Thus, the stationary distribution follows a Poisson distribution with rate λ/μ.

Exercise 4.
a. The routing equations are $\gamma_1 = 1 + 3/4\, \gamma_2$, $\gamma_2 = p\gamma_1$, and $\gamma_3 = (1-p)\gamma_1$. This gives $\gamma_1 = 1/(1-\frac{3}{4}p)$, $\gamma_2 = p/(1-\frac{3}{4}p)$, and $\gamma_3 = (1-p)/(1-\frac{3}{4}p)$. For stability, we need to find p such that $\gamma_1 < 2$, as queue 1 is the bottleneck. Hence, we need $p < 2/3$ for stability. This is a Jackson network and the stationary distribution is thus of product form:

$$
\begin{aligned}
\pi(n_1, n_2, n_3) &= \left(1-\frac{\gamma_1}{2}\right)\left(\frac{\gamma_1}{2}\right)^{n_1}\left(1-\frac{\gamma_2}{2}\right)\left(\frac{\gamma_2}{2}\right)^{n_2}\left(1-\frac{\gamma_3}{2}\right)\left(\frac{\gamma_3}{2}\right)^{n_3} \\
&= \left(1-\frac{1}{2-3/2\,p}\right)\left(\frac{1}{2-3/2\,p}\right)^{n_1}\left(1-\frac{p}{2-3/2\,p}\right)\left(\frac{p}{2-3/2\,p}\right)^{n_2} \\
&\quad \times \left(1-\frac{1-p}{2-3/2\,p}\right)\left(\frac{1-p}{2-3/2\,p}\right)^{n_3}.
\end{aligned}
$$

Exercise 5.
a. The first two terms correspond to the situations to whom the vaccine is sold: (i) $\mathbb{E}[\min\{D,S\}]$ is the expected sales to the government, making a profit of $p - 0.2p = 0.8p$, and (ii) $\mathbb{E}(S-D)^+$ is the expected sales elsewhere, making a profit of $0.1p - 0.2p = -0.1p$. The final term is the fixed production costs (the variable production costs are incorporated in the profits).

Note that $\mathbb{E}\min(D,S) = \mathbb{E}D - \mathbb{E}(D-S)^+$. To maximize $P(S)$, we need to minimize $0.8p\mathbb{E}(D-S)^+ + 0.1p\mathbb{E}(S-D)^+$, which corresponds to a newsvendor. Hence, using standard arguments,

$$
S^* = F_D^{-1}\left(\frac{0.8p}{0.8p+0.1p}\right) = F_D^{-1}\left(\frac{8}{9}\right).
$$

b. Conditioning of the demand from the government D, we have

$$
\begin{aligned}
\mathbb{E}(D-S)^+ &= \int_S^{50} \frac{1}{50}(x-S)\mathrm{d}x \\
&= \frac{1}{50}\left(\frac{50^2}{2} - 50S + \frac{S^2}{2}\right) = \frac{(50-S)^2}{100}.
\end{aligned}
$$

The fraction of lost sales is thus

$$
\frac{\mathbb{E}(D-S)^+}{\mathbb{E}D} = \left(\frac{50-S}{50}\right)^2.
$$