Exam Applied Stochastic Modeling
16 December 2019, 8:45-11:30 hours

This exam consists of **4** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times$ $9/30 + 1$.

Exercise 1. Consider an M/M/1 priority queue with two types of customers. The arrival rates are 1 and 2 for types 1 and 2, respectively. The service times are exponentially distributed with rate μ for both types. Type 1 has priority over type 2.
a. [3 pt.] Determine the expected waiting time of both customer types. Argue what happens with the expected waiting time of type 2 when $\mu \downarrow 3$.
b. [4 pt.] To protect type 1 customers, type 2 customers are only admitted to the system when the system is empty, and rejected otherwise. Derive the stationary distribution of the number of customers in the system.
c. [2 pt.] Suppose that type 1 customers are also (just as type 2 customers) only admitted when the system is empty, and rejected otherwise. Give the blocking probability.

Exercise 2. Consider a machine-repairshop with N machines in total. The lifetime of a machine is exponentially distributed with rate λ. When a machine goes down, it needs to be repaired. The repair facility has a single repairman, and the repair times follow an exponential distribution with rate 1.
a. [3 pt.] Let X_t be the number of machines in repair at time t and let Y_t be the number of working machines at time t. Draw the state diagram with the transition rates for the process X_t. Also, draw the state diagram with the transition rates for the process Y_t. Which state description is more convenient for determining stationary distributions?

It turns out that not all type of repairs can be carried out by the same facility. When a machine goes down, it needs to be repaired at facility 1 with probability $2/3$ or at facility 2 with probability $1/3$. Both repair facilities have a single repairman, and the repair times follow exponential distributions with rates 1 and 2, respectively.
b. [3 pt.] Model the new machine-repairshop as a closed queueing network and formulate the routing equations. Give the stationary distribution of the joint number of machines at each location upto the normalizing constant (i.e. you do not need to provide the normalizing constant).
c. [2 pt.] Assume that $N = 1$. Give regeneration epochs and use renewal theory to derive the long-run fraction of time the machine is working.

Exercise 3. The royal palace is opened to the public at time epoch 2 and attracts many visitors. Already before time 2 visitors arrive who wait in front of the gate. In particular, between 0 and 2 visitors arrive according to a time-dependent Poisson process with rate $50t$ for $t \in [0, 2]$. Between 2 and 4 visitors arrive according to a Poisson process with rate 100. After time 4 no new visitors are allowed.

Visitors arriving before time 2 are only allowed to enter the royal palace at time 2 (and wait until the royal palace is opened). Each visitor spends exactly two hours in the royal palace (not counting any time in front of the gate).
a. [3 pt.] What is the expected number of visitors that wait in front of the gate at time 2? And what is the corresponding distribution? Argue that the number of arriving visitors between 2 and 4 follows a Poisson distribution with rate 200.
b. [4 pt.] Determine the expected number of visitors $m(\tau)$ in the royal palace at time τ and verify that $m(\tau) = 100(\tau - 1)$ for $\tau \in [2, 4]$ and $m(\tau) = 100(6 - \tau)$ for $\tau \in (4, 6]$. Make a sketch of $m(\tau)$ and explain what happens with the expected number of visitors at time 4.

For safety reasons, there is a maximum number of tickets that can be sold. Tickets between 0 and 2 cost 10 euro, whereas tickets between 2 and 4 cost 15 euro. The royal palace aims to maximize the revenue and should decide on the number of tickets S they should hold back until the moment that the royal palace opens.
c. [2 pt.] Use marginal arguments to verify that the optimal value for S that maximizes the revenue is the largest S that satisfies

$$\sum_{k=0}^{S-1} e^{-200} \frac{200^k}{k!} \leq \frac{1}{3}.$$

Exercise 4. Consider a continuous-review continuous-product deterministic inventory model with holding costs, order costs and zero lead time (the EOQ model). Demand arrives continuously at rate λ per time unit. The holding costs are h per item per time unit. The total order costs depend on the order size. Specifically, when the order is Q, then the order costs are $K - \alpha Q$. The maximum order size is K/α.
a. [2 pt.] Use renewal theory to derive that the total average costs per time unit for order size $Q \in (0, K/\alpha]$ are

$$C(Q) = \frac{\lambda K}{Q} - \alpha\lambda + \frac{1}{2} h Q.$$

b. [2 pt.] Determine the optimal order level Q^*.