Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

December 16, 2019, 8:45 - 11:30 hours

Exercise 1.

a. As the service times of both customer types follow the same (exponential) distribution, it follows directly that $\mathbb{E}S_1 = \mathbb{E}S_2 = \mathbb{E}S = {}^1\!/\!_\mu$ and $\mathbb{E}S^2 = {}^2\!/\!_{\mu^2}$. Consequently, the residual service time upon arrival reads

$$\mathbb{E}R = \frac{\lambda \mathbb{E}S^2}{2} = \frac{3 \times 2}{2 \times \mu^2} = \frac{3}{\mu^2}.$$

Now, using the loads $\rho_1 = {}^1\!/\!_\mu$, $\rho_2 = {}^2\!/\!_\mu$, and combining the above, we obtain

$$\begin{aligned}
\mathbb{E}W_Q(1) &= \frac{\mathbb{E}R}{1-\rho_1} = \frac{3/\mu^2}{1 - {}^1\!/\!_\mu} = \frac{3}{\mu(\mu-1)} \\
\mathbb{E}W_Q(2) &= \frac{\mathbb{E}R}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{3/\mu^2}{(1-{}^1\!/\!_\mu)(1-{}^1\!/\!_\mu-{}^2\!/\!_\mu)} = \frac{3}{(\mu-1)(\mu-3)}.
\end{aligned}$$

When $\mu \downarrow 3$, then $\mathbb{E}W_Q(2) \to \infty$, as the total load tends to 1 (i.e. $\mu > 3$ is required for stability of the system, as $\rho_1 + \rho_2 = {}^3\!/\!_\mu$).

b. Let $X(t)$ denote the total number of customers at time t. The transition diagram of the birth-and-death process $X(t)_{t\geq 0}$ is given in Figure 1. The distribution of the number of customers in the system follows from the balance equations (for sets): $3\pi(0) = \mu\pi(1)$ and $1 \times \pi(i-1) = \mu\pi(i)$, for $i = 2, 3, \ldots$. The first equation yields $\pi(1) = {}^3\!/\!_\mu\,\pi(0)$, whereas the combination gives

$$\pi(i) = \left(\frac{1}{\mu}\right)^{i-1} \pi(1) = 3\left(\frac{1}{\mu}\right)^{i} \pi(0), \qquad i = 1, 2, \ldots.$$

Using normalization (assuming $\mu > 1$ for stability), we obtain that

$$\pi(0) = \left[1 + \sum_{i=1}^{\infty} 3\left(\frac{1}{\mu}\right)^i\right]^{-1} = \left[1 + \frac{3/\mu}{1 - {}^1\!/\!_\mu}\right]^{-1} = \frac{\mu - 1}{\mu + 2}.$$

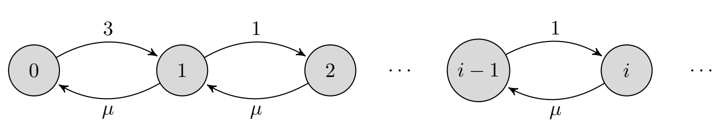


Figure 1: State diagram Exercise 1b.

c. Observe that the model now corresponds to an M/M/1/1 queue (or Erlang loss model) with arrival rate 3 and service rate μ. Due to PASTA, the blocking probability equals

$\pi(1) = \frac{3}{\mu+3}$. Note that $\pi(1)$ can either be obtained from solving the appropriate Markov chain with state space $\{0, 1\}$, or by writing out $B(1, {}^{3}\!/\!_{\mu})$ of the Erlang loss model.

Exercise 2.

a. The transition diagram for the process $X(t)_{t\geq 0}$ (number of machines in repair) is given in Figure 2; the transition diagram for the process $Y(t)_{t\geq 0}$ (number of working machines) is given in Figure 3. Note that $X(t)_{t\geq 0}$ corresponds to an Engset delay model and $Y(t)_{t\geq 0}$ to an Erlang loss model with arrival rate 1 and service rate λ. The process $Y(t)_{t\geq 0}$ seems simpler when solving directly for the stationary distribution, but it may be argued otherwise by refering to the Engset model.

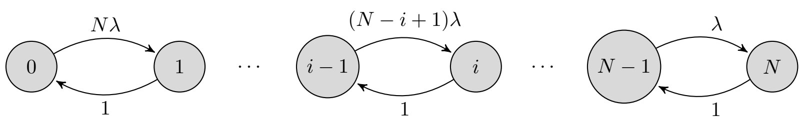


Figure 2: State diagram of $X(t)_{t\geq 0}$ (number of machines in repair) for Exercise 2a.

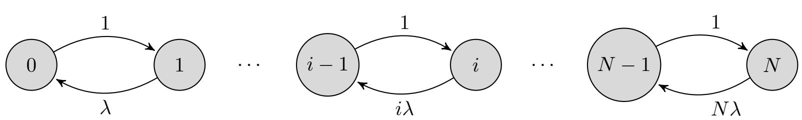


Figure 3: State diagram of $Y(t)_{t\geq 0}$ (number of working machines) for Exercise 2a.

b. Let $N_i(t)$ denote the number of machines at facility i at time t. Then $(Y(t), N_1(t), N_2(t))_{t\geq 0}$ constitutes a closed Jackson network (we will refer to 1 as the node of working machines and 2 and 3 will refer to facility 1 and 2, respectively). The routing equations are: $\gamma_1 = \gamma_2 + \gamma_3$, $\gamma_2 = {}^{2}\!/\!_{3}\gamma_1$, and $\gamma_3 = {}^{1}\!/\!_{3}\gamma_1$. Thus, the solution has the following form:

$$\pi(n_1, n_2, n_3) = C \times \frac{({}^{\gamma_1}\!/\!_{\lambda})^{n_1}}{n_1!} e^{-\gamma_1/\lambda} \times \gamma_2^{n_2} \times \left(\frac{\gamma_3}{2}\right)^{n_3},$$

with C the normalizing constant. For instance, choosing $\gamma_2 = 1$ yields $\gamma_1 = {}^{3}\!/\!_{2}$ and $\gamma_3 = {}^{1}\!/\!_{2}$. Thus,

$$\pi(n_1, n_2, n_3) = C \times \frac{({}^{3}\!/\!_{2\lambda})^{n_1}}{n_1!} e^{-3/2\lambda} \times \left(\frac{1}{4}\right)^{n_3},$$

c. We define the moments that the machine is just repaired (and thus as good as new) as the regeneration epochs. We have that the expected cycle length is $\mathbb{E}T = \frac{1}{\lambda} + \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{1}{\lambda} + \frac{5}{6}$. Using the renewal reward theorem, the long-run fraction of time the machine is working equals

$$\mathbb{P}(\text{working}) = \frac{\mathbb{E}[\text{working per cycle}]}{\mathbb{E}T} = \frac{{}^{1}\!/\!_{\lambda}}{{}^{1}\!/\!_{\lambda} + {}^{5}\!/\!_{6}} = \frac{6}{6 + 5\lambda}.$$

Exercise 3.

a. The number of visitors waiting at the gate at time 2 equals the number of arrivals during [0,2]. Hence, the expected number of visitors at the gate at time 2 then is $\int_0^2 50t\,dt = \frac{50}{2} t^2 \big|_{t=0}^{2} = 100$. As visitors arrive according to a (time-dependent) Poisson process, it follows directly that the number of visitors at the gate at time 2 follows a Poisson distribution with rate 100 (use the first definition).

Similarly, the number of arriving visitors in [2,4] follows a Poisson distibution with rate $\int_2^4 100\mathrm{d}t = 200$ (use again the first definition).

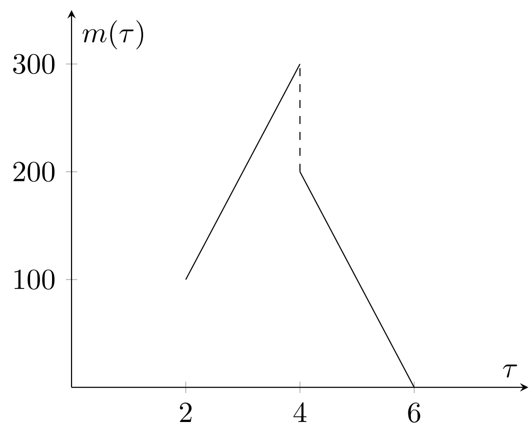
b. The expected number of visitors in the royal palace $m(\tau)$, for $\tau \in [2,4)$, consists of the expected number of arriving visitors during $[2,\tau]$ and the expected number of visitors that were waiting in front of the gate when the royal palace opened:

$$m(\tau) = \int_2^\tau 100\mathrm{d}t + 100 = 100(\tau - 2) + 100 = 100(\tau - 1).$$

For $\tau \in [4,6)$, the expected number of visitors in the royal palace equals the expected number of arriving visitors in $[\tau - 2, 4]$. Thus,

$$m(\tau) = \int_{\tau-2}^4 100\mathrm{d}t = 100t|_{t=\tau-2}^4 = 100(4 - (\tau - 2)) = 100(6 - \tau).$$

Please see the figure below for a sketch of $m(\tau)$. At time epoch 4, there is a big drop (expected value of 100) in the number of visitors at the royal palace. These are the visitors that were waiting in front of the gate at time 2; they all finish their visit after exactly 2 hours, i.e., at time epoch 4.



c. This example is similar to the airline revenue management example. For marginal arguments, it needs to be decided whether each subsequent ticket should be hold back, or should be sold for 10 euro's. When the S-th ticket is hold back, the expected revenue is $15 \times \mathbb{P}(D \geq S)$, with D the demand during $[2,4]$. Thus, the optimal S is the largest S that satisfies $15 \times \mathbb{P}(D \geq S) \geq 10$. From part a, it follows that D follows a Poisson distribution with rate 200, such that $\mathbb{P}(D \geq S) = 1 - \sum_{k=0}^{S-1} e^{-200} \frac{200^k}{k!}$. Some rewriting provides the desired result, i.e., the optimal S is the largest S that satisfies

$$\sum_{k=0}^{S-1} e^{-200} \frac{200^k}{k!} \leq \frac{15 - 10}{15} = \frac{1}{3}.$$

Exercise 4.

a. A natural choice for the regeneration epochs are the moments of replenishment, i.e., the moment that the inventory becomes 0 (a sketch may be convenient). Just as in the regular EOQ, the cycle length T is ${}^{Q}\!/\!{}_{\lambda}$. Now, the holding cost per cycle are $\frac{1}{2}QTh$ (just as in the regular EOQ). The order cost per cycle are $K - \alpha Q$. Using the renewal reward theorem, it follows directly that the long-run average cost per time unit are

$$C(Q) = \frac{\text{cost per cycle}}{T} = \frac{K - \alpha Q}{Q/\lambda} + \frac{1}{2}hQ = \frac{\lambda K}{Q} - \alpha\lambda + \frac{1}{2}hQ.$$

b. To find the optimal order level, take derivatives with respect to Q: $\frac{\mathrm{d}}{\mathrm{d}Q}C(Q) = -\frac{K\lambda}{Q^2} + \frac{1}{2}h$. Setting the derivative to 0 yields $\frac{1}{2}h = \frac{K\lambda}{Q^2}$, hence $Q^* = \sqrt{2K\lambda/h}$, as only the positive solution is required.

(Remark 1: It is easy to verify that this is an optimum by taking the second derivative of $C(Q)$.)

(Remark 2: Note that Q^* may be larger than K/α, in which case only K/α can be ordered.)