Exam Applied Stochastic Modeling
17 December 2018, 8:45-11:30 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times$ 9/33 + 1.

Exercise 1. Consider an M/G/1 queue with arrival rate $\lambda \in [0,1)$. The service times of customers are uniformly distributed between 0 and 2. Customers with a service requirement of at most 1 are considered to be small and receive (non-preemptive) priority over customers with a service requirement that are larger than 1.
a. [3 pt.] Show that the expected waiting time of small customers ($\mathbb{E}W_Q(1)$) and the expected waiting time of large customers ($\mathbb{E}W_Q(2)$) are:

$$\mathbb{E}W_Q(1) = \frac{2\lambda/3}{1-\lambda/4}, \qquad \text{and} \qquad \mathbb{E}W_Q(2) = \frac{2\lambda/3}{(1-\lambda/4)(1-\lambda)}.$$

b. [2 pt.] Make a sketch of $\mathbb{E}W_Q(1)$ and $\mathbb{E}W_Q(2)$ as a function of $\lambda \in [0,1)$ and explain its behavior for $\lambda \to 1$.

Exercise 2. Consider a birth-and-death process with death rate μ in all states $1, 2, \ldots$, birth rate 4 in state 0 and birth rate α in states $1, 2, \ldots$; this may represent the number of customers in the system for a type of single-server queue.
a. [1 pt.] For which values of α and μ is this process stable?
b. [4 pt.] Draw the state diagram with transition rates for this birth-and-death process and determine the distribution of the number of customers in the system.
c. [2 pt.] Assume that $\alpha = 0$. Give regeneration epochs and use renewal theory to determine the probability that the system is empty.
d. [3 pt.] Assume again that $\alpha = 0$. However, if the idle time is at least t, then the subsequent service time is exponentially distributed with rate 2μ (the service remains exponential with rate μ if the idle time was smaller than or equal to t). Determine the probability that the system is empty.

Exercise 3. A service facility opens at time 0. After time 0, customers arrive to an infinite-server system according to a homogeneous Poisson process with rate 10. The service times follow an exponential distribution with rate 1.
a. [3 pt.] Use the thinning properties of Poisson processes to show that the number of customers in the system at time $\tau \geq 0$ follows a Poisson distribution with rate $10\,(1 - e^{-\tau})$.
b. [3 pt.] Consider some time $\tau \geq 2$. If at time τ a customer is found present and the customer is already in the system for at least 2 time units, it is considered to be big. Show that the number of big customers in the system at time $\tau \geq 2$ follows a Poisson distribution with rate $m_{\text{big}}(\tau) = 10\left(e^{-2} - e^{-\tau}\right)$.
c. [1 pt.] Make a sketch of $m_{\text{big}}(\tau)$ for $\tau \geq 2$ and explain its behavior.

Exercise 4. An open queueing network consists of two queues, both having a single server. New customers arrive to queue 1 according to a Poisson process with rate λ. After being served at queue 1, customers go back to queue 1 with probability p_1, go to queue 2 with probability p_2 and leave the system with probability p_3 (with $p_i \in [0,1]$ and $\sum_{i=1}^{3} p_i = 1$). After being served at queue 2, customers leave the network. The service times at both queues are exponential with rate 3.
a. [3 pt.] Formulate the routing equations for this system. For which values of λ and p_i is the system stable? Also give the stationary distribution of the joint number of customers.

Assume that $p_1 = p_3 = 0$ and $p_2 = 1$. When station 1 becomes empty, the total service capacity is allocated to station 2. In this situation the departure rate at station 2 (if non empty) is temporally increased to 6 (also in case of a single job we here assume that both servers may process this job). The server switches back to station 1 as soon as a job arrives there.
b. [3 pt.] Draw the state diagram with corresponding transition rates for the two-dimensional Markov process of the number of customers at both stations. Give the balance equations for states (n_1, n_2) with $n_1, n_2 > 0$ and $(0, n_2)$ with $n_2 > 0$, where n_i denotes the number of customers in queue i.

Exercise 5. An organization of artists has created K similar works of art. The works of art can be sold to a trader for p_2 per item or the organization can try to sell them themselves for $p_1 > p_2$. The trader is willing to buy all items that the organization is willing to sell them; the demand when the organization tries to sell them themselves is according to a random variable D. Works of art that the organization is not able to sell can be offered at an auction and yields a revenue of $v < p_2$ per item.
a. [1 pt.] Let S be the number of items that the organization is trying to sell themselves. Argue that the expected income is

$$P(S) = p_2(K - S) + p_1 \mathbb{E} \min(D, S) + v \mathbb{E}(S - D)^+.$$

b. [3 pt.] Give the policy that maximizes the expected income of the organization. How should it be taken into account that the demand is discrete?
c. [1 pt.] How many works of art should be sold to the trader when $v > p_2$?