Exam Applied Stochastic Modeling
29 March 2018, 18:30-21:15 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points $\times\, ^9/_{33} + 1$.

Exercise 1. At a science center, a popular exposition is open during the interval $[0, 8]$ (representing 9:00 - 17:00 hours). During the interval $[0, 6]$ visitors arrive according to an inhomogeneous Poisson process with rate λe^{-t} at time $t \in [0, 6]$. After time 6, no new visitors arrive. For simplicity, we assume that visitors arrive one at a time. We also assume that a visit takes exactly 2 hours.
a. [2 pt.] Determine the expected number of *new* visitors that arrive during $[0, 1]$ and the expected number of *new* visitors that arrive during $[5, 6]$.
b. [3 pt.] Determine $m(\tau)$, i.e., the mean number of visitors at the exposition at time τ, for $\tau \in [0, 8]$. What is the probability that at time 7 the science center is already empty?
c. [2 pt.] Make a sketch of $m(\tau)$ as a function of $\tau \in [0, 8]$ and explain its behavior. Also, argue when the peak in $m(\tau)$ occurs.

Exercise 2. Consider a queueing system with 2 queues in parallel and a Poisson arrival process with rate $^2/_3$. Customers are routed to one of the two queues directly upon arrival (and leave after receiving service). Each queue has a single server and an infinite waiting buffer. There are two types of customers. A customer is of type 1 with probability $\alpha \in [0, 1]$, and of type 2 otherwise. The service times of type 1 are exponential with rate α and the service times of type 2 are exponential with rate $1 - \alpha$.
a. [1 pt.] Verify that the mean service time of an arbitrary customer is $\mathbb{E}S = 2$.
b. [3 pt.] Suppose that the routing of both customers types is completely random (i.e. a customer is assigned to queue 1 with probability 0.5, independent of all other customers). Determine the expected waiting time for an arbitrary customer.
c. [3 pt.] Suppose that type 1 is routed to queue 1 and type 2 is routed to queue 2. Determine the expected waiting time for both types and the expected waiting time for an arbitrary customer.
d. [2 pt.] Make a sketch of the expected waiting times of an arbitrary customer as a function of $\alpha \in [0, 1]$ for the routing mechanisms in parts a and b and explain its behavior.

Exercise 3. An open queueing network consists of two queues, both having a single server. New customers arrive to queue 2 according to a Poisson process with rate λ. After being served at queue 1, customers go to queue 2 with probability $2/3$ and go back to queue 1 with probability $1/3$. After being served at queue 2, customers leave the network with probability p and go to queue 1 with probability $1-p$. The service times at queue 1 are exponential with rate 1 and at queue 2 exponential with rate 2.
a. [2 pt.] Formulate the routing equations for this system. For which values of λ and p is the system stable?
b. [2 pt.] Give the balance equation for state (n_1, n_2) with $n_1, n_2 > 0$ and n_i denoting the number of customers in queue i.

Exercise 4. Consider the open queueing network of Exercise 3. It is decided to transform the open network into a closed network. Specifically, assume that there are N customers in the network, whereas $\lambda = 0$ and $p = 0$.
a. [4 pt.] Draw the state diagram with the transition rates for the number of customers at queue 1 and determine the distribution of the queue length at queue 1.
b. [3 pt.] Assume that $N = 1$. We are interested in the probability that the customer is at queue 1. Give regeneration epochs and use renewal theory to determine the probability that the customer is at queue 1.

Exercise 5. A company tries to find the optimal balance in fixed and flexible labor workforce. To do so, the company considers a single period of time. The fixed workforce is S employees and has to be determined in advance. The demand for labor during this single period is according to a random variable D. When the demand exceeds the fixed workforce, the company hires flexible labor from an external company. The fixed workforce costs 100 per employee, whereas flexible labor costs 150 per employee.
a. [4 pt.] Give the total workforce costs for such a single period as a function of S, and give an expression for the occupancy of the fixed workforce (i.e. the expected fraction of fixed workforce for which there is demand). Determine the optimal fixed workforce S^* that minimizes the total costs.
b. [2 pt.] Suppose that the demand D follows a normal distribution with mean μ and standard deviation σ. What is the impact on S^* when the variability in demand σ increases, whereas the mean μ remains the same? And what would S^* be when $\sigma \downarrow 0$?