Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

December 18, 2017, 8:45 - 11:30 hours

Exercise 1.

a. The transition diagram for the M/M/∞ queue is given in Figure 1. The distribution of the equilibrium number of customers follows from the balance equations (for sets) $\lambda\pi(i-1) = i\mu\pi(i)$. Expressing in terms of $\pi(0)$ yields $\pi(i) = (\lambda/\mu)^i/(i!)\pi(0)$. Due to normalization, it is required that $\pi(0)\sum_{i=0}^{\infty}(\lambda/\mu)^i/(i!) = 1$, giving $\pi(0) = e^{-\lambda/\mu}$. Hence, the equilibrium number of customers follows a Poisson distribution with rate λ/μ.

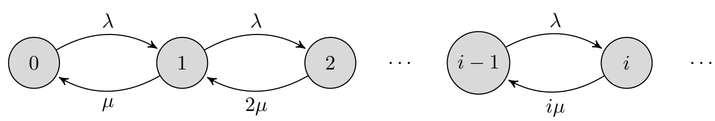

Figure 1: State diagram Exercise 1a.

b. The mean number of customers at time 0 is given by

$$m(0) = \int_{-\infty}^{0} \lambda e^{-\mu(-t)}\mathrm{d}t = \int_{0}^{\infty} \lambda e^{-\mu u}\mathrm{d}u = \frac{\lambda}{\mu},$$

where the second step follows from the substitution $u = -t$. Note that this result can also be intuitively verified as it corresponds to the mean number of customers in the M/M/∞ queue being in steady state (and corresponds with part a). For $\tau > 0$, we have

$$m(\tau) = \int_{-\infty}^{0} \lambda e^{-\mu(\tau-t)}\mathrm{d}t = \lambda e^{-\mu\tau} \int_{-\infty}^{0} e^{\mu t}\mathrm{d}t = \frac{\lambda}{\mu}e^{-\mu\tau},$$

where the last step follows from the same computation as for $m(0)$.

Please see Figure 2 for a sketch of $m(\tau)$. The figure displays that the queue gradually drains after closing (as there are no new arrivals).

Exercise 2.

a. The aggregate average waiting is smallest when priority is given to customers with short service times (in line with e.g. SJF). Hence, the best order is 1, 2, 3, where 1 has highest and 3 has lowest priority.

b. For the mean service time of an arbitrary customer, we have

$$\mathbb{E}S = \frac{3}{7} \times 1 + \frac{3}{7} \times 2 + \frac{1}{7} \times 3 = \frac{12}{7}.$$

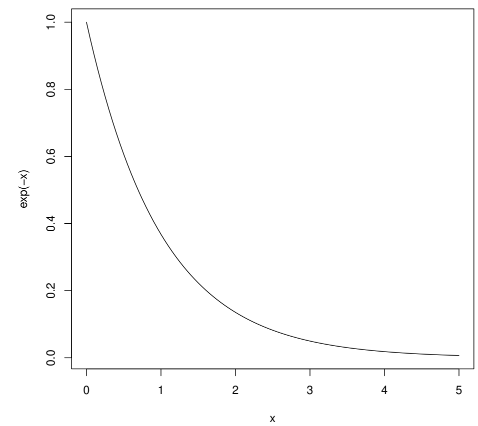

Figure 2: For exercise 1b, sketch of $m(\tau)$ as a function of τ (for $\lambda = \mu = 1$).

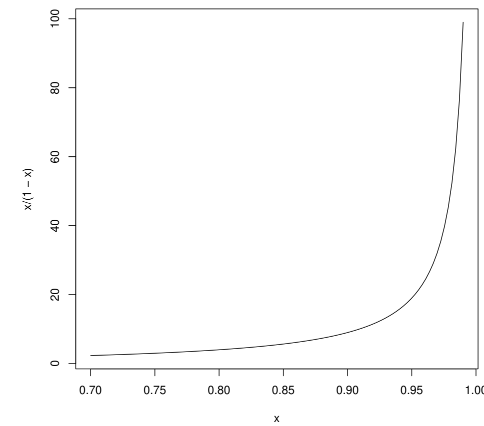

Figure 3: For exercise 3b, sketch of $\mathbb{E}W_Q^{(2)}$ as a function of p.

Similarly, the second moment of an arbitrary customer is

$$\mathbb{E}S^2 = \frac{3}{7} \times 1^2 + \frac{3}{7} \times 2^2 + \frac{1}{7} \times 3^2 = \frac{24}{7}.$$

Consequently, the residual service time upon arrival reads

$$\mathbb{E}R = \frac{\lambda \mathbb{E}S^2}{2} = \frac{1}{2} \times \frac{7}{18} \times \frac{24}{7} = \frac{2}{3}.$$

Now, using the loads $\rho_1 = 1/6$, $\rho_2 = 1/3$, and $\rho_3 = 1/6$ and combining the above, we obtain

$$\begin{aligned} \mathbb{E}W_Q(1) &= \frac{\mathbb{E}R}{1-\rho_1} = \frac{4}{5} \\ \mathbb{E}W_Q(2) &= \frac{\mathbb{E}R}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{8}{5} \\ \mathbb{E}W_Q(3) &= \frac{\mathbb{E}R}{(1-\rho_1-\rho_2)(1-\rho_1-\rho_2-\rho_3)} = 4 \end{aligned}$$

c. The average service time of class 2 remains the same, so it does not affect the order. It does affect the waiting time of all classes, as the additional variability influences the residual service time R. Following similar lines as in part b, we now have

$$\mathbb{E}S^2 = \frac{3}{7} \times 1^2 + \frac{3}{7} \times \frac{2}{(1/2)^2} + \frac{1}{7} \times 3^2 = \frac{36}{7},$$

and thus the new $\mathbb{E}R = \frac{\lambda \mathbb{E}S^2}{2} = \frac{1}{2} \times \frac{7}{18} \times \frac{36}{7} = 1$. This yields the following mean waiting time of class 1 (note that ρ_i remains the same):

$$\mathbb{E}W_Q(1) = \frac{\mathbb{E}R}{1-\rho_1} = \frac{6}{5}.$$

Exercise 3.
a. The routing equations are $\gamma_1 = \lambda$ and $\gamma_2 = p\gamma_1$, and hence $\gamma_2 = p\lambda$. The system is stable if $\lambda < 2$ and $p\lambda < 1$, thus $\lambda < \min\{2, {}^1\!/\!_p\}$.
b. This is a Jackson network and the stationary distribution is thus of product form:

$$\pi(n_1, n_2) = \left(1 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^{n_1} (1-p)\,p^{n_2} = \left(\frac{1}{2}\right)^{n_1+1} (1-p)\,p^{n_2}.$$

Using results for the M/M/1 queue (with load p and service rate 1), the expected waiting time for queue 2 is $\mathbb{E}W_Q^{(2)} = \frac{p}{1-p}$. See Figure 3 for a sketch. When $p \uparrow 1$, it holds that the load $\rho \uparrow 1$ and queue 2 becomes critically loaded. As a result, the waiting time will 'explode'.
c. Let X_t be the number of customers at queue 1 at time t. Then, $(X_t)_{t\geq 0}$ is a birth-death process with state diagram and transition rates given in Figure 4. Using balance equations, we obtain $\lambda\pi(0) = 2\pi(1)$ and $\lambda\pi(i-1) = 3\pi(i)$, for $i = 2, 3, \ldots$. Hence, $\pi(1) = {}^\lambda\!/\!_2\, \pi(0)$ and $\pi(i) = {}^\lambda\!/\!_3\, \pi(i-1) = ({}^\lambda\!/\!_3)^{i-1}\, \pi(1) = {}^3\!/\!_2({}^\lambda\!/\!_3)^i\, \pi(0)$. Using normalization, we have

$$\pi(0)\left[1 + \sum_{i=1}^{\infty} \frac{3}{2}\left(\frac{\lambda}{3}\right)^i\right] = 1,$$

giving, after some calculus, $\pi(0) = (1 - {}^\lambda\!/\!_3)/(1 + {}^\lambda\!/\!_6)$. Thus, for $i = 1, 2, \ldots$,

$$\pi(i) = \frac{3}{2}\left(\frac{\lambda}{3}\right)^i \frac{1 - {}^\lambda\!/\!_3}{1 + {}^\lambda\!/\!_6}.$$

Remark: In part c, it would also be possible to take $\lambda = 1$.

Due to Burke's output theorem, the output process of queue 1 is still a Poisson process with rate λ. Thus, this has no impact on queue 2.

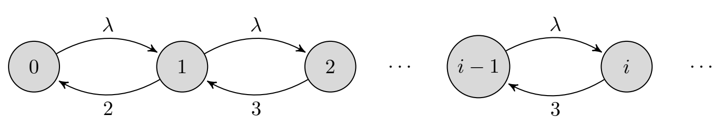

Figure 4: State diagram Exercise 3c.

Exercise 4.
a. Regeneration epochs are the moments that the team returns from the break (or, alternatively, when they are just starting a break). The fraction of time that the team is working is $\mathbb{E}$[time working during a cycle]/$\mathbb{E}$[cycle time]. Now, the term $\mathbb{E}$[time working during a cycle] can be obtained by conditioning on the duration of the first project:

$$\begin{aligned}\mathbb{E}[\text{time working during a cycle}] &= \int_0^2 \left(x + \frac{1}{\mu}\right)\mu e^{-\mu x}\mathrm{d}x + \int_2^\infty x \times \mu e^{-\mu x}\mathrm{d}x \\ &= \frac{1}{\mu}\int_0^2 \mu e^{-\mu x}\mathrm{d}x + \int_0^\infty x \times \mu e^{-\mu x}\mathrm{d}x \\ &= \frac{1}{\mu}\left(1 - e^{-2\mu}\right) + \frac{1}{\mu} \;=\; \frac{1}{\mu}\left(2 - e^{-2\mu}\right).\end{aligned}$$

Clearly, $\mathbb{E}[\text{cycle time}] = (2 - e^{-2\mu})/\mu + {}^1\!/_2$. Combining the above, it holds that the fraction of time that the team is working (defined as $f(\mu)$) is

$$f(\mu) = \frac{(2 - e^{-2\mu})/\mu}{(2 - e^{-2\mu})/\mu + {}^1\!/_2} = \frac{2 - e^{-2\mu}}{2 - e^{-2\mu} + \mu/2}.$$

b. First, $\lim_{\mu \downarrow 0} f(\mu) = 1$. When $\mu \downarrow 0$, a project takes excessively long, so the time for a break is negligible compared to the time for a project.
Second, $\lim_{\mu \to \infty} f(\mu) = 0$. When $\mu \to \infty$, the (two) projects are very short and the time between two successive breaks becomes negligible. The team thus spends (almost) all time on having breaks.

Exercise 5.
a. In terms of the notation of Koole, this concerns an EOQ model with $K = 18$, $h = 2$, and $\lambda = 2$. The optimal order size is $Q^* = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{2\times18\times2}{2}} = 6$. The reorder level is 2 and the order costs are $C(Q^*) = \sqrt{2K\lambda h} = \sqrt{2 \times 18 \times 2 \times 2} = 12$.
b. Please make a sketch of the sample path of the inventory level. A natural choice for the regeneration epochs are the moments of replenishment. For order size Q, the time to clear the inventory after replenishment is ${}^Q\!/_2$, as demand arrives at rate $\lambda = 2$. Due to the lead time of $L = 1$, the total time between successive replenishments (and thus the cycle length) is ${}^Q\!/_2 + 1$. Now, there are the following cost components:
Order costs per cycle: 18
Holding costs per cycle: $2 \times \frac{1}{2}Q \times {}^Q\!/_2 = Q^2/2$
Costs for lost sales per cycle: $2 \times 1 \times q$
Combining the above and using the renewal reward theorem, the long-run average costs per time unit are

$$\frac{1}{1 + Q/2}\left(18 + Q^2/2 + 2q\right).$$