Exam Applied Stochastic Modeling
15 February 2017, 18:30-21:15 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points/4 + 1.

Exercise 1. An art exposition is open during the interval $[0, 4]$, corresponding to a four-hour evening. During the first two hours, visitors arrive according to a Poisson process with rate λ per hour. After two hours, the doors are closed and visitors can no longer enter. For simplicity, we assume that visitors arrive one at a time. We also assume that a visit takes exactly 2 hours.
a. [2 pt.] Give the probability that the first arrival occurs after time t, for every $t \in [0, 4]$. Also give the probability that during the second hour only one visitor arrives.
b. [3 pt.] Determine $m(\tau)$, i.e., the mean number of visitors at the exposition at time τ, for $\tau \in [0, 4]$.
c. [2 pt.] Make a sketch of $m(\tau)$ as a function of $\tau \in [0, 4]$ and explain its behavior. When is the peak in $m(\tau)$? Can this be explained?

Exercise 2. Consider the M/M/1/N queue with 1 server and N places in the system.
a. [4 pt.] Draw the state diagram with the transition rates and determine the distribution of the queue length. What is the probability that an arriving customer is rejected?
b. [2 pt.] Assume that $N = 1$. Give regeneration epochs and use renewal theory to determine the probability that the system is empty.

Exercise 3. Consider an M/G/1 queue with arrival rate $1/3$. The service time of customers consists of either one or two stages. The first stage of service takes exactly 1 time unit. With probability $p \in (0, 1]$ there is a second stage of service, taking exactly $1/p$ time units; with probability $1 - p$ service is completed after the first stage. We assume that it is known upon arrival how many stages of service are required for a customer.
a. [4 pt.] Let S be the service time of an arbitrary customer. Show that $\mathbb{E}S = 2$ and $\mathbb{E}S^2 = 3 + \frac{1}{p}$, and calculate the expected waiting time in case customers are served in the order of arrival (FCFS). Describe and explain how the expected waiting time behaves as a function of $p \in (0, 1]$.

b. [3 pt.] Suppose that customers requiring one stage of service have (nonpreemptive) priority over customers requiring two stages of service. Calculate the expected waiting time for customers requiring one stage of service, and explain the two different ways in which p affects this expected waiting time.

Exercise 4. An open queueing network consists of two queues. New customers arrive to queue 1 according to a Poisson process with rate λ. After being served at queue 1, customers always go to queue 2. After being served at queue 2, customer go back to queue 1 with probability $p \in [0, 1]$ or leave the system with probability $1 - p$. The service times at queue 1 is exponential with rate 1 and at queue 2 exponential with rate 2. Queue 1 has an infinite number of servers, whereas queue 2 has a single server.
a. [2 pt.] Formulate the routing equations for this system. For which values of λ and p is the system stable?
b. [3 pt.] Give the stationary distribution of the joint number of customers, and the expected sojourn time per queue.

It is decided to transform the open network into a closed network. Specifically, assume that there are 3 customers in the network, whereas $\lambda = 0$ and $p = 1$.
c. [3 pt.] Model the number of customers at queue 1 as a birth-and-death process and derive its marginal stationary distribution.

Exercise 5. Consider a continuous-review continuous-product deterministic inventory model with holding costs, order costs and zero lead time (the EOQ model). Demand arrives continuously at rate 1 per time unit. The order costs are K per order and management wants to consider two different holding cost functions when the inventory level is x: (i) linear holding costs $h_1(x) = h_1 x$, and (ii) quadratic holding costs $h_2(x) = h_2 x^2$. The order quantity is denoted by Q.
a. [3 pt.] Argue that the average order cost per time unit are K/Q. Consider case (i), i.e. linear holding costs, and suppose that at time 0 an order of Q is placed. Then, the inventory level at time $t \in [0, Q]$ equals $Q - t$. Use this, in combination with renewal theory, to derive the average holding costs per time unit.
b. [2 pt.] Suppose that the quadratic holding costs are used, i.e. consider case (ii). Use renewal theory again for the holding costs, to show that the average costs per time unit are given by

$$C_2(Q) = \frac{K}{Q} + h_2 \frac{Q^2}{3}.$$

c. [3 pt.] Derive the optimal order quantity based on cost function $C_2(Q)$ and show that the corresponding Q^* is an optimum.