Exam Applied Stochastic Modeling
8 February 2016

This exam consists of **6** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points/4 + 1.

Exercise 1. The arrival process of customers is modeled as an inhomogeneous Poisson process. Between times 0 and 2 the arrival rate is λ per time unit, between times 2 and 4 the arrival rate is 3λ per time unit and after time 4 the arrival rate is 2λ per time unit.
a. [2 pt.] Assume that it is now time 0. Give the probability that the first arrival occurs after time t, for every $t > 0$.
b. [3 pt.] Customers remain in the system for exactly 3 time units. Show that the number of customers present at time 5 has a Poisson distribution and determine the parameter.

Exercise 2. The lifetime of a critical spare part has an exponential distribution with rate μ. The spare part is replaced by a new one upon failure or upon reaching the critical age T, whichever occurs first. A cost f is incurred for each failure replacement and a cost of c for each preventive replacement. The lifetimes of the spare parts are independent of each other.
a. [2 pt.] Specify regeneration epochs and indicate how renewal theory can be used to determine the long-run average cost per time unit.
b. [3 pt.] Show that the long-run average cost per time unit under the age-replacement rule equals

$$\frac{\mu\left(f + (c - f)e^{-\mu T}\right)}{1 - e^{-\mu T}}.$$

Exercise 3. Consider a web server farm consisting of two servers working according to FCFS. Arrivals follow a Poisson process with rate 1/3. There are small and large jobs. The probability that a job is large is p, whereas it is small with probability $1 - p$, with $p \in (0, \frac{1}{2}]$. The size of large jobs is $\frac{2}{p}$ and the size of small jobs is $\frac{2}{1-p}$.
a. [3 pt.] Suppose jobs are randomly assigned to one of the servers and let S be the service time of an arbitrary job. Show that $\mathbb{E}S = 4$ and $\mathbb{E}S^2 = \frac{4}{p(1-p)}$, and calculate the expected waiting time.
b. [3 pt.] Suppose that size-interval splitting is used such that large jobs are routed to server one and small jobs are routed to server two. Calculate the expected waiting times of large and small jobs and determine the weighted average waiting time (of an arbitrary job).
c. [2 pt.] Compare the results of questions a and b for the cases $p = \frac{1}{2}$ and $p \to 0$ and explain what you observe.

Exercise 4. Consider a repairshop with 2 repairman and 3 machines (the machine repairman model). The lifetime of a machine follows an exponential distribution with rate $\frac{1}{2}$. When a machine goes down, it is repaired by a repairman when available or waits until a repairman is available otherwise. A machine can be repaired by only one repairman simultaneously. The repairtime of a machine is exponentially distributed with rate 1, after which it is as good as new.
a. [3 pt.] Model this as a birth-and-death process. Calculate the stationary distribution of the number of machines that are down.
b. [3 pt.] Suppose that all three machines are down. We are interested in the clearing time T until all three machines that are currently down are repaired. Show that the Laplace-Stieltjes transform of this clearing time T is

$$\mathbb{E}\left[e^{-sT}\right] = \left(\frac{2}{2+s}\right)^2 \frac{1}{1+s}.$$

Use this transform to determine the mean of T.

Exercise 5. An open queueing network consists of three queues each with a single server. New customers arrive to queues 1 and 2, where the external arrival rate for both queues is λ. After being served at queue 1, customers go to queue 2 with probability $\frac{1}{3}$; with probability $\frac{2}{3}$ they go to queue 3. After being served at queue 2, they go to queue 1 with probability $\frac{1}{2}$; with probability $\frac{1}{2}$ they go to queue 3. After being served at queue 3, customers leave the system. The service times at the three queues are exponential with rate 2.
a. [3 pt.] Formulate the routing equations for this system. For which value of λ is the system stable?
b. [2 pt.] Determine the expected total number of customers in the system in terms of λ.

Exercise 6. A production process in a factory yields waste that is temporarily stored on the factory site. Waste accumulates lineairly at rate λ. A company to remove waste can be called at any moment (continuous review), after which they are assumed to arrive instantaneously and remove all waste. The factory uses the following control rule: when the amount of waste reaches level Q, a call is made to remove the waste. There are holding costs h per unit waste per time unit. There are fixed costs V per time that waste is removed and variable costs v per unit of waste that is removed.
a. [2 pt.] Argue that the total average costs per time unit are

$$C(Q) = v\lambda + \frac{\lambda V}{Q} + \frac{hQ}{2}.$$

b. [3 pt.] Determine the optimal level Q^* for removing waste.
c. [2 pt.] Assume that there is a lead time L between making the call to remove waste and the moment the actual waste is removed. We assume that L is a positive random variable. The control rule still is that a call is made when waste reaches level Q. Determine the new cost function.