Exam Applied Stochastic Modeling
14 December 2015

This exam consists of **6** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points/$4 + 1$.

Exercise 1. The arrival process of customers is modeled as an inhomogeneous Poisson process. Between times 0 and 3 the arrival rate is 2λ per time unit and after time 3 the arrival rate is λ per time unit.

a. [2 pt.] Assume that it is now time 0. Give the probability that the first arrival occurs after time t, for every $t > 0$.

b. [3 pt.] Customers remain in the system for exactly 5 time units. Show that the number of customers present at time 5 has a Poisson distribution and determine the parameter.

Exercise 2. Consider the M/G/1/1 system with 1 server and no queue.

a. [2 pt.] Assume that service time are exponentially distributed. Use a continuous-time Markov chain to determine the probability that the system is empty and give the blocking probability.

b. [3 pt.] Assume general service times. Give regeneration epochs and use renewal theory to determine the probability that the system is empty.

Exercise 3. Consider an M/G/1 queue with arrival rate 1/2. Customer require either a 'regular' service or a 'special' service. The probability that a customer requires a regular service is 2/3, whereas the regular service time is equal to 1. The probability that a customer requires a special service is 1/3, and the special service takes an exponentially distributed amount of time with rate 1/2.

a. [2 pt.] Let S be the service time of an arbitrary customer. Show that $\mathbb{E}S = 4/3$ and $\mathbb{E}S^2 = 10/3$, and calculate the expected waiting time.

b. [3 pt.] Calculate the expected waiting time for 'regular' and 'special' service when 'regular' service tasks have (nonpreemptive) priority.

c. [2 pt.] Explain what happens with the weighted-average waiting time compared to the answer of question a.

Exercise 4. Consider a queue with a single server. Potential customers arrive according to a Poisson proces with rate 2. When there are customers waiting upon arrival, the arriving customer joins the queue with probability 1/2 (and leaves immediately with probability 1/2). Service times are exponential with rate 3/2.
a. [4 pt.] Model this as a birth-and-death process. Calculate the stationary distribution of the number of customers in the system.
b. [4 pt.] The Laplace-Stieltjes transform of the waiting time W_Q is

$$\mathbb{E}\left[e^{-sW_Q}\right] = \frac{1}{5} + \frac{2}{5}\frac{1}{1/2+s}.$$

Use this transform to determine the mean waiting time. What is its distribution?

Exercise 5. An open queueing network consists of two queues in tandem with each a single server. New customers arrive to queue 1 according to a Poisson process with rate 1. The service time at queue i is exponential with rate μ_i, $i = 1, 2$.
a. [1 pt.] For which values of μ_1 and μ_2 is the system stable?
b. [2 pt.] Give the stationary distribution of the joint number of customers in terms of μ_1 and μ_2, and show that the total expected waiting time is

$$\frac{1/\mu_1}{\mu_1 - 1} + \frac{1/\mu_2}{\mu_2 - 1}.$$

c. [2 pt.] Assume that the total service capacity $\mu_1 + \mu_2$ is equal to 4, but you may choose how to divide it over the two queues. For which values of μ_1 and μ_2 will the total expected waiting time be minimized. Give also an intuitive explanation for your answer.

Exercise 6. A shop sells goods where inventory is controlled using a periodic-review continuous-product inventory model with lost sales lead time 0. The demand during a day is a random variable D. At the end of the day, an order is placed that arrives at the beginning of the next day (thus, lead time 0). The order is such that the inventory level after the order arrives is S, that is S is the order up to level. There are holding costs h per item and costs r per lost sale. Holding costs are determined based on the average inventory level during a day, which is determined as the average of the inventory at the beginning and the inventory at the end of the day.
a. [2 pt.] Argue that the cost function as a function of S is

$$C(S) = \frac{h}{2}S + \frac{h}{2}\mathbb{E}(S-D)^+ + r\mathbb{E}(D-S)^+.$$

b. [4 pt.] Determine the optimal order up to level S^* that minimizes the cost function, assuming $r \geq h/2$. What is the optimal order up to level S^* in case $r < h/2$? Give an intuitive explanation.