Faculty of Science
Department of Mathematics, VU

Final Applied Analysis: Financial Mathematics
31-05-2023, 15:30–17:45

Use of calculator, phone, laptop, book or notes is not allowed.
The exam consists of 4 questions on two pages.
A formula sheet on two pages is provided at the end of the exam.
Please write the calculations and arguments leading to your answers.
Motivate your answers.

Points

1a: 3	2a: 3	3a: 4	4a: 3
b: 3	b: 3	b: 3	b: 3
	c: 4		c: 4
			d: 3

$$\text{Grade} = \frac{\#\text{points}}{4} + 1$$

1. Consider a put option contract that has the following form: it gives you the right to sell a share for a fixed exercise price E at several pre-determined dates between now and an expiry date T (you are also allowed to exercise at T). Denote the value of this put option by $P(S,t)$; denote also the values of the American and European put options by $P_{Am}(S,t)$ and $P_{Eu}(S,t)$, respectively.

a. What inequality exists between these three put option prices?

b. Obviously, a similar kind of call option can be considered: it gives you the right to buy a share for a fixed price E at several pre-determined dates between now and an expiry date T, including T itself. Under which condition do we have equality for the three kinds of call option: this one, the European one and the American one?

a. Clearly $P_{Eu}(S,t) \leq P(S,t) \leq P_{Am}(S,t)$ (does not need any argumentation, you get progressively more rights, so you have to pay more). **3 points, presumably all or nothing.**

b. Since the American call option and the European call option are equal when there is no dividend payment during the lifetime of the option **1point**, and since we have a similar inequality for the call options as for the put options **1 point** the prices will coincided when there is no dividend payment **1 point.**

2. A *perpetual American put* option gives the holder the right to sell a share at any time in the future for a fixed exercise price E. (So, this is in fact an American put with $T = \infty$.) Denote the value of the option by $V(S,t)$.

It can be shown that this option is not explicitly dependent on t, so $\frac{\partial V}{\partial t} = 0$. Hence we can actually write $V(S)$ instead of $V(S,t)$.

a. Give the ordinary differential equation which $V(S)$ should satisfy before it is exercised. What is the value of $V(S)$ once it is exercised?

b. The point where it is optimal to exercise is denoted (as usual) by S_f. This will also not depend on t of course. What continuity conditions should be satisfied in the point S_f for the perpetual American put option?

c. Consider the function

$$V(S) = \begin{cases} E - S & \text{for } S < S_f \\ (E - S_f)\left(\frac{S}{S_f}\right)^{-2r/\sigma^2} & \text{for } S > S_f \end{cases}$$

where $S_f = \frac{2rE}{2r+\sigma^2}$.

Show that this function satisfies the differential equation you found in part **a.** for $S > S_f$, and satisfies the continuity conditions you found in part **b**. Also show that it satisfies $\lim_{S\to\infty} V(S) = 0$, which is the boundary condition at infinity. (You may assume that the risk-less interest rate r is positive.)

a. Since the partial derivative with respect to time is zero, and since any option before being exercised or before expiry satisfies the Black-Scholes equation **1 point**, we have that this option satisfies the ordinary differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

1 point Also, after exercise, the value is $E - S$ (as we will not exercise when the value S is bigger than E) **1 point**.

b. The continuity conditions are the same as the one from the ordinary American put **1 point**, so V is continuous in S_f **1 point** and $V'(S_f) = -1$ **1 point**. Alternatively the latter can be phrased as V' is continuous, or as Δ is continuous, or as $\Delta = -1$ in the point S_f.

c. For $S > S_f$ we have $rSV'(S) = -\frac{2r^2}{\sigma^2}(E - S_f)\left(\frac{S}{S_f}\right)^{-2r/\sigma^2}$ **1/2 point**, and

$$\tfrac{1}{2}\sigma^2 S^2 V''(S) = \tfrac{1}{2}\sigma^2 \cdot \tfrac{-2r}{\sigma^2} \cdot \left(\tfrac{-2r}{\sigma^2} - 1\right)(E - S_f)\left(\tfrac{S}{S_f}\right)^{-2r/\sigma^2}$$
$$= r\left(\tfrac{2r}{\sigma^2} + 1\right)(E - S_f)\left(\tfrac{S}{S_f}\right)^{-2r/\sigma^2}.$$ **1 point**

(This involves a combination of S and $\frac{1}{S_f}$ to $\frac{S}{S_f}$ in both derivatives.)

Inserting into the differential equation we see that this is indeed a solution **1 point**.

Clearly V is continuous in $S = S_f$, and $V'(S_f) = -\frac{2r}{\sigma^2 S_f}(E - S_f) = -\frac{2r}{\sigma^2}(\frac{E}{S_f} - 1)$.

Insert the formula for S_f to see that this is minus one **1 point**.

That the limit for S to infinity is zero follows from $r/\sigma^2 > 0$ **1/2 point**.

3. Given is a function $u(x,\tau)$ which satisfies the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$$

and with intial condition $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$.

a. Write $u(x,\tau) = e^{\alpha x + \beta \tau} v(x,\tau)$. Determine α and β in terms of c such that $v(x,\tau)$ satisfies the heat equation.

b. Find an integral expression for $u(x,\tau)$.

a. We have

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= e^{\alpha x + \beta \tau}\left(\beta v(x,\tau) + \frac{\partial v}{\partial \tau}\right), \\
\frac{\partial u}{\partial x} &= e^{\alpha x + \beta \tau}\left(\alpha v(x,\tau) + \frac{\partial v}{\partial x}\right), \\
\frac{\partial^2 u}{\partial x^2} &= e^{\alpha x + \beta \tau}\left(\alpha^2 v(x,\tau) + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right).
\end{aligned}$$

2 points Inserting in the partial differential equation for u, we obtain after dividing by the exponential term

$$\beta v + \frac{\partial v}{\partial \tau} = \alpha^2 v(x,\tau) + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + c\left(\alpha v(x,\tau) + \frac{\partial v}{\partial x}\right).$$

1 point Since v has to satisfy the heat equation, we see that α and β are determined from $\beta = \alpha^2 + c\alpha$ and $2\alpha + c = 0$. So, $\alpha = -\frac{c}{2}$ and $\beta = \frac{c^2}{4} - \frac{c^2}{2} = -\frac{c^2}{4}$.
1 point

b. Use the formula for $v(x,\tau)$ from the formula sheet to obtain:

$$u(x,\tau) = \frac{e^{-(cx/2 + c^2\tau/4)}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v_0(s) e^{-(x-s)^2/4\tau} ds$$

1 1/2 points. Now it remains to replace $v_0(s) = e^{-\alpha s} u_0(s)$ to arrive at the final formula

$$u(x,\tau) = \frac{e^{-(cx/2 + c^2\tau/4)}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{cs/2} u_0(s) e^{-(x-s)^2/4\tau} ds.$$

1 1/2 points. This may be rewritten as

$$u(x,\tau) = \frac{e^{-c^2\tau/4}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{c(s-x)/2 - (x-s)^2/4\tau} ds.$$

4.a Use Taylor's theorem to show that

$$f''(x) = \frac{f(x+2h) + f(x+h) - 4f(x) + f(x-h) + f(x-2h)}{5h^2} + O(h^2)$$

b. Use this to derive the following finite difference approximation for the heat equation

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{u_{n+2}^m + u_{n+1}^m - 4u_n^m + u_{n-1}^m + u_{n-2}^m}{5(\delta x)^2}.$$

Introduce $\alpha = \frac{\delta\tau}{5(\delta x)^2}$ (pay attention, this is different from what you are used to). Rewrite the above equation as an explicit finite difference method, expressing u_n^{m+1} explicitly.

c. Now assume errors are of the form $e_n^m = \lambda^m \sin(n\omega)$. You may assume that these errors satisfy the same equation you found in part **b**. Show that $\lambda = 1 - \alpha(4 - 2\cos(\omega) - 2\cos(2\omega))$.

d. Show that the method is stable when $\frac{\delta\tau}{(\delta x)^2} < \frac{8}{5}$ and unstable when $\frac{\delta\tau}{(\delta x)^2} > \frac{8}{5}$. You may make use of the fact that $0 \le 4 - 2\cos(\omega) - 2\cos(2\omega)) \le 6\frac{1}{4}$ as is easily verified by elementary calculus (but you do not have to show this).

a. We have

$$\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8}{6}f'''(x) + O(h^4),\\
f(x+h) &= f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}f'''(x) + O(h^4),\\
f(x-h) &= f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}f'''(x) + O(h^4),\\
f(x-2h) &= f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{8}{6}f'''(x) + O(h^4).
\end{aligned}$$

2 points. Add these and subtract $4f(x)$ left and right to arrive at

$$f(x+2h) + f(x+h) - 4f(x) + f(x-h) + f(x-2h) = 5h^2 f''(x) + O(h^4).$$

Divide by $5h^2$ to get the result **1 point.**

b. Denote $u(n\delta x, m\delta\tau)$ by u_n^m, and use the forward difference approximation for $\frac{\partial u}{\partial \tau}$ and the approximation from part a for the second order partial derivative with respect to x. That gives the formula in the exercise **1 1/2 point.** Now multiply by $\delta\tau$ and move the term u_n^m from the left to the right, to obtain

$$u_n^{m+1} = \alpha u_{n+2}^m + \alpha u_{n+1}^m + (1-4\alpha)u_n^m + \alpha u_{n-1}^m + \alpha u_{n-2}^m \qquad \textbf{1 1/2 point}$$

c. Use that e_n^m satisfies the same equation as the u_n^m and insert to get

$$\lambda^{m+1}\sin(n\omega) = \lambda^m\left(\alpha\sin((n+2)\omega) + \alpha\sin((n+1)\omega) + (1-4\alpha)\sin(n\omega) + \alpha\sin((n-1)\omega) + \alpha\sin((n-2)\omega)\right).$$

1 point. Divide by λ^m **1/2 point** and use the formula for $\sin(a+b)$ four times **1 point**. Several terms cancel, and then we are left with

$$\lambda\sin(n\omega) = \sin(n\omega)\left(1 + \alpha(-4 + 2\cos(\omega) + 2\cos(2\omega))\right).$$

1 point. Divide by $\sin(n\omega)$ to get the desired formula **1/2 point**.

d. First note that $\lambda \le 1$ for all $\alpha > 0$ since $4 - 2\cos(\omega) - 2\cos(2\omega) \ge 0$. So the method is stable (errors will decay), when $\lambda > -1$ **1 point**. Knowing that $4 - 2\cos(\omega) - 2\cos(2\omega) \le 6\frac{1}{4} = \frac{25}{4}$ we see that $\lambda \ge -1$ for all ω when $1 - \frac{25}{4}\alpha > -1$ **1/2 point**. That is the case when $\frac{25}{4}\alpha < 2$, equivalently, $\alpha < \frac{8}{25}$ **1/2 point**. Recall that $\alpha = \frac{\delta\tau}{5(\delta x)^2}$, so the method is stable when $\frac{\delta\tau}{(\delta x)^2} < \frac{8}{5}$ **1/2 point**. Instability will occur when there is a value of ω for which $\lambda < -1$, which is the case when $\frac{\delta\tau}{(\delta x)^2} > \frac{8}{5}$ by the same reasoning **1/2 point**.

Formulas

Binomial tree related

Replicating portfolio $V = \Delta S + \Pi$, where $\Delta = \frac{V_u - V_d}{S_u - S_d}$.
Value of V as an expected value

$$V_n = e^{-r\delta t}(qV_u + (1-q)V_d), \qquad q = \frac{e^{r\delta t}S_n - S_d}{S_u - S_d}$$

Model for stock price

$$dS = S(\mu dt + \sigma dX)$$

where dX is normally distributed with expectation zero and variance dt.
S itself then has a lognormal distribution.

Functions of S and Itô's lemma: use Taylor's theorem and replacing dX^2 by dt

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt.$$

Payoff functions

For call and put:

$$C(S,T) = \max(S-E,0), P(S,T) = \max(E-S,0).$$

Asset or nothing: S if $S > E$, and 0 whenvener $S < E$.
Cash or nothing: E if $S > E$, and 0 whenvener $S < E$.

Black Scholes equation

Derived by considering $V = \Delta S + \Pi$; using Itô's lemma and remembering that Π should be a bond, we derive

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = r(V - S\frac{\partial V}{\partial S}).$$

With proper initial and boundary conditions we derive values for the call and put options.

Formulas for call and put option

$$\begin{aligned}
C(S,t) &= SN(d_1) - Ee^{-r(T-t)}N(d_2),\\
P(S,t) &= Ee^{-r(T-t)}N(-d_2) - SN(-d_1),\\
d_1 &= \frac{\log\frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma(\sqrt{T-t})},\\
d_2 &= \frac{\log\frac{S}{E} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma(\sqrt{T-t})}.
\end{aligned}$$

Heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

General solution with initial condition $u(x,0) = u_0(x)$

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-(x-s)^2/(4\tau)}\, ds.$$

Reduction of Black Scholes equation to heat equation

$S = Ee^x$, $\tau = (T-t)\frac{1}{2}\sigma^2$.
$v(x,\tau) = \frac{1}{E}V(S,t)$, $u(x,\tau) = e^{-(\alpha x + \beta\tau)} v(x,\tau)$,
where $\alpha = -\frac{1}{2}(k-1)$, $\beta = -\frac{1}{4}(k+1)^2$. Then $v(x,\tau)$ satisfies the heat equation when $V(S,t)$ satisfies the Black Scholes equation.

Taylor's Theorem

When f is $n+1$ times differentiable then

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots + \frac{1}{n!}h^n f^{(n)}(x) + \frac{1}{(n+1)!}h^{n+1} f^{(n+1)}(x+\theta),$$

where $x+\theta$ is between x and $x+h$.

Goniometric identities

$$\begin{aligned}
\sin(a+b) &= \sin a \cos b + \cos a \sin b, \\
\sin(-a) &= -\sin a, \quad \cos(-a) = \cos a, \\
\cos(2a) &= 2\cos^2 a - 1 = 1 - 2\sin^2 a, \\
\sin(2a) &= 2 \sin a \cos a.
\end{aligned}$$