Faculty of Sciences
Department Mathematics, Vrije Universiteit

Applied Analysis/Financial Mathematics
24-05-2019, 12:00–14:45

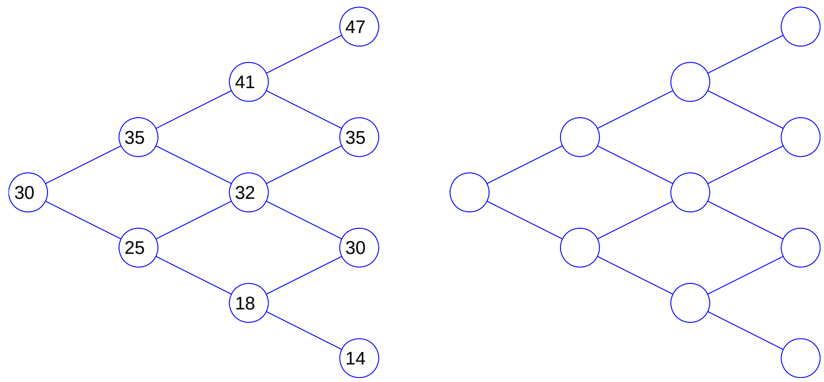
Instructions

1. You are allowed to use a calculator during this exam.
2. Please write your name on the front page of the exam *before* you answer the questions. Write your name on every sheet of paper you hand in.
3. This exam consists of 5 questions
4. Answering instruction
   (a) Write your answers on the paper provided.
   (b) If you are asked to present an argument for your answer or to explain some issue, write your answers in correct sentences.
5. Your grade will be calculated based on the total points scored: it is the number of points divided by 3 plus 1. The number of points per question is specified in the table below.

question	a	b	c	d	total
1.	1	2	2	1	6
2.	2	3			5
3.	2	2			4
4.	3	$1\frac{1}{2}$	$1\frac{1}{2}$		6
5.	$1\frac{1}{2}$	3	$1\frac{1}{2}$		6
total					27

6. Your **mobile phone should be switched off** and should be put in your bag. Your bag needs to be closed. Any other electronic device, such as Ipads, laptops etc. should also be switched off and in your bag.

1. Given is the stock price in a binomial tree as follows:



You may assume in this exercise that the interest rate $r = 0$.

a. Determine the martingale probabilities q for every fork of the tree.

b. Consider the cash or nothing option, paying out a fixed amount of $E = 30$ when $S \geq E = 30$ at time level three, and nothing when $S < E$. Determine the price of the option at every node in the tree.

c. For all nodes at time level two, determine the replicating portfolio, i.e. determine the values of Δ and Π.

d. Explain how the structure of the replicating portfolio will change for the bottom node at time level two when E changes from 30 to any value higher than 30.

2. Consider the European call option $C(S, t, T, E)$.

a. Show by a no-arbitrage argument that for $T_1 > T_2$ one has $C(S, t, T_1, E) > C(S, t, T_2, E)$.

b. Show that $\frac{\partial C}{\partial T} > 0$ by computing the partial derivative explicitly. You may use that we proved in class that $SN'(d_1) = Ee^{-r(T-t)}N'(d_2)$.

3. Given is the option $V(S, t)$ which pays out at expiry date T the value of the stock S when $S < E$ and a fixed amount E when $S \geq E$.

a. Show how $V(S, T)$ can be expressed in terms of S and the payoff $C(S, T)$ of the European call option, and sketch the payoff function.

b. Give an explicit formula for the value $V(S, t)$ of the option for all S and t in terms of S, E, t, T, r and σ, and justify your answer.

4. Consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x, 0) = f(x)$. In this exercise we shall consider a numerical method to solve this equation. As usual, consider a grid with stepsize δx in the x direction, $\delta\tau$ in the τ direction, and denote $u(n\delta x, m\delta\tau)$ by u_n^m.

a. Use the backward difference for the τ derivative, the symmetric central difference for the second order derivative in x and the central difference for the first derivative in x to derive a formula expressing u_n^m in terms of $u_{n-1}^{m+1}, u_n^{m+1}, u_{n+1}^{m+1}$. Use $\alpha = \frac{\delta\tau}{(\delta x)^2}$, $\beta = \frac{\delta\tau}{2\delta x}$ and $\gamma = \delta\tau$.

b. How should we take u_n^0?

c. Discuss stability of the method.

5. For $c \geq 0$ consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x, 0) = f(x)$.

Assume that $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to\pm\infty} f'(x)$ exist and that f is twice continuously differentiable.

For $c > 0$ consider the following function

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + c\tau - 2\sqrt{\tau} z) e^{-z^2}\, dz.$$

In this exercise you will show that this function satisfies the partial differential equation.

a. Assuming that differentiation (both with respect to τ and with respect to x) and integration may be interchanged, check that

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + c\tau - 2\sqrt{\tau} z) e^{-z^2}\, dz,$$
$$\frac{\partial u}{\partial \tau} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + c\tau - 2\sqrt{\tau} z) \left(c - \frac{1}{\sqrt{\tau}} z\right) e^{-z^2}\, dz.$$

b. Derive a similar formula for $\frac{\partial^2 u}{\partial x^2}$, and use that formula and integration by parts (*partiële integratie*) to see that the function $u(x,\tau)$ satisfies the equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$.

c. Assuming that you may interchange $\lim_{\tau\downarrow 0}$ and the integral, show that indeed $u(x, 0) = f(x)$.