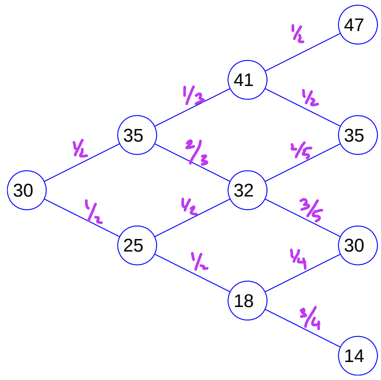
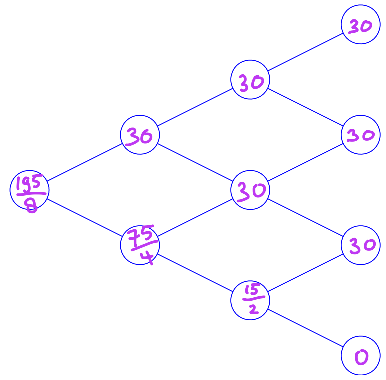
1. Given is the stock price in a binomial tree as follows:





You may assume in this exercise that the interest rate $r = 0$.

a. Determine the martingale probabilities q for every fork of the tree.

b. Consider the cash or nothing option, paying out a fixed amount of $E = 30$ when $S \geq E = 30$ at time level three, and nothing when $S < E$. Determine the price of the option at every node in the tree.

c. For all nodes at time level two, determine the replicating portfolio, i.e. determine the values of Δ and Π.

d. Explain how the structure of the replicating portfolio will change for the bottom node at time level two when E changes from 30 to any value higher than 30.

(a) for $r=0$ we have $q = \dfrac{S_0 - S_d}{S_u - S_d}$ The values are indicated in the diagram above

(b) $V_0 = q V_u + (1-q) S_d$ at every node, starting at $t=3$,

where the payoff is 30 if $S > 30$ and 0 if $S < 30$.

The values are given in the diagram above

leading to the option price $V = \frac{195}{8}$ at $t=0$.

(c) $\Delta = \dfrac{V_u - V_d}{S_u - S_d}$ and $\Pi = V - \Delta S$ At $t=2$ this gives $\Delta = 0$ $\Pi = 30$

$\Delta = 0$ $\Pi = 30$

$\Delta = \frac{15}{8}$ $\Pi = -\frac{105}{4}$ for the replicating portfolio.

(d) Then $\Delta = 0$ and $\Pi = 0$.

2. Consider the European call option $C(S,t,T,E)$.

a. Show by a no-arbitrage argument that for $T_1 > T_2$ one has $C(S,t,T_1,E) > C(S,t,T_2,E)$.

b. Show that $\frac{\partial C}{\partial T} > 0$ by computing the partial derivative explicitly. You may use that we proved in class that $SN'(d_1) = Ee^{-r(T-t)}N'(d_2)$.

(a) A lot of variants can be used, but here is one no-arbitrage argument to show that

$C(S,t,T_1,E) > C(S,t,T_2,E)$ for $T_1 > T_2$ and $t \leq T_2$ (the second option makes no sense for $t > T_2$)

Consider the portfolio $V(S,t,T_1,T_2,E) = C(S,t,T_1,E) - C(S,t,T_2,E)$.

Evaluating at $t = T_2$ we find $V(S,T_2,T_1,T_2,E) = C(S,T_2,T_1,E) - C(S,T_2,T_2,E) = C(S,T_2,T_1,E) - (S-E)_+$

Since we know for all option that $C(S,t,T,E) > (S-E)_+$ for all $t < T$, we have, taking $t = T_2 < T_1$:

$$V(S,T_2,T_1,T_2,E) = C(S,T_2,T_1,E) - (S-E)_+ > 0 \quad \text{for } T_2 < T_1$$ without risk.

Hence the standard no-arbitrage-opportunity argument now implies that

$V(S,t,T_1,T_2,E) > 0$ for all $t \leq T_2$. $\Rightarrow C(S,t,T_1,E) - C(S,t,T_2,E) > 0$ for $t \leq T_2 < T_1$

$\Rightarrow C(S,t,T_1,E) > C(S,t,T_2,E)$ for $t \leq T_2 < T_1$.

(b) $C(S,t,T,E) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$ with $N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-s^2/2}\,ds$ and $d_1 = \frac{\log\frac{S}{E} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$d_2 = \frac{\log\frac{S}{E} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

$$\frac{\partial C}{\partial T} = SN'(d_1)\frac{\partial d_1}{\partial T} + rEe^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial T}$$

$$= Ee^{-r(T-t)}N'(d_2)\frac{\partial d_1}{\partial T} + rEe^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial T}$$

$$= Ee^{-r(T-t)}N'(d_2)\left[\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T}\right] + rEe^{-r(T-t)}N(d_2)$$

$$= Ee^{-r(T-t)}N'(d_2)\left[N'(d_2)\frac{\sigma}{2\sqrt{T-t}} + rN(d_2)\right] > 0$$

$$\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T} = \frac{\partial[d_1 - d_2]}{\partial T} = \frac{\sigma}{2\sqrt{T-t}}$$

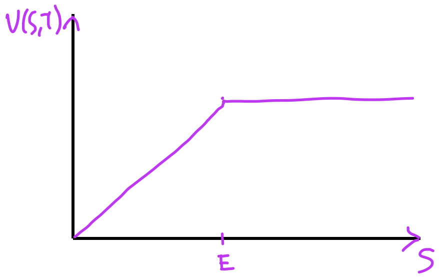
since $N(d) > 0$ and $N'(d) > 0$ (it is a cumulative probability density)

3. Given is the option $V(S,t)$ which pays out at expiry date T the value of the stock S when $S < E$ and a fixed amount E when $S \geq E$.

a. Show how $V(S,T)$ can be expressed in terms of S and the payoff $C(S,T)$ of the European call option, and sketch the payoff function.

b. Give an explicit formula for the value $V(S,t)$ of the option for all S and t in terms of S, E, t, T, r and σ, and justify your answer.

a



$$V(S,T) = S - C(S,T)$$

b Since the pay-off is $V(S,T) = S - C(S,T)$ the reproducing portfolio is one stock and minus one call. Hence (by no-arbitrage) the value is

$$V(S,t) = S - C(S,t) = S - \left[S N(d_1) - E e^{r(T-t)} N(d_2)\right]$$
$$= S N(-d_1) + E e^{r(T-t)} N(d_2)$$

where d_1 and d_2 are given by the formulas in question 2b.

4. Consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x,0) = f(x)$. In this exercise we shall consider a numerical method to solve this equation. As usual, consider a grid with stepsize δx in the x direction, $\delta\tau$ in the τ direction, and denote $u(n\delta x, m\delta\tau)$ by u_n^m.

a. Use the backward difference for the τ derivative, the symmetric central difference for the second order derivative in x and the central difference for the first derivative in x to derive a formula expressing u_n^m in terms of $u_{n-1}^{m+1}, u_n^{m+1}, u_{n+1}^{m+1}$. Use $\alpha = \frac{\delta\tau}{(\delta x)^2}$, $\beta = \frac{\delta\tau}{2\delta x}$ and $\gamma = \delta\tau$.

b. How should we take u_n^0?

c. Discuss stability of the method.

(a) $\frac{\partial u}{\partial \tau}(m\delta x, n\delta\tau) \approx \frac{u_n^m - u_n^{m-1}}{\delta\tau}$ $\qquad u(m\delta x, n\delta\tau) \approx u_n^m$

$\frac{\partial u}{\partial x}(m\delta x, n\delta\tau) \approx \frac{u_{n+1}^m - u_{n-1}^m}{2\delta x}$

$\frac{\partial^2 u}{\partial x^2}(m\delta x, n\delta\tau) \approx \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{(\delta x)^2}$

We find $\frac{u_n^m - u_n^{m-1}}{\delta\tau} = \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{(\delta x)^2} - \frac{u_{n+1}^m - u_{n-1}^m}{2\delta x} + u_n^m$ for the discretisation scheme of the PDE.

Replacing m by m+1 and multiplying by $\delta\tau$ we find

$$u_n^{m+1} - u_n^m = \alpha\left[u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\right] - \beta\left[u_{n+1}^{m+1} - u_{n-1}^{m+1}\right] + \gamma u_n^m$$

Hence $u_n^m = (-\alpha-\beta)\,u_{n-1}^{m+1} + (1+2\alpha-\gamma)\,u_n^{m+1} + (-\alpha+\beta)\,u_{n+1}^{m+1}$

(b) $u_n^0 = u(n\delta x, 0) = f(n\delta x)$.

(c) $\beta = 2\alpha\,\delta x$ and $\gamma = \alpha(\delta x)^2$ and since δx is tiny we may neglect the effect of β and γ compared to α.
But then the resulting scheme is the implicit scheme discussed in the course, which we saw is stable for any $\alpha > 0$.

5. For $c \geq 0$ consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x,0) = f(x)$.

Assume that $\lim_{x\to\pm\infty} f(x)$ and $\lim_{x\to\pm\infty} f'(x)$ exist and that f is twice continuously differentiable.

For $c > 0$ consider the following function

$$u(x,\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + c\tau - 2\sqrt{\tau}z)e^{-z^2}\, dz.$$

In this exercise you will show that this function satisfies the partial differential equation.

a. Assuming that differentiation (both with respect to τ and with respect to x) and integration may be interchanged, check that

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + c\tau - 2\sqrt{\tau}z)e^{-z^2}\, dz,$$
$$\frac{\partial u}{\partial \tau} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x + c\tau - 2\sqrt{\tau}z)\left(c - \frac{1}{\sqrt{\tau}}z\right)e^{-z^2}\, dz.$$

b. Derive a similar formula for $\frac{\partial^2 u}{\partial x^2}$, and use that formula and integration by parts (*partiële integratie*) to see that the function $u(x,\tau)$ satisfies the equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$.

c. Assuming that you may interchange $\lim_{\tau\downarrow 0}$ and the integral, show that indeed $u(x,0) = f(x)$.

(a) $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x+c\tau-2\sqrt{\tau}z)e^{-z^2}\,dz$ by just differentiating the integrand

By the chain rule:

$$\frac{\partial u}{\partial \tau} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f'(x+c\tau-2\sqrt{\tau}z)\left[c - \frac{z}{\sqrt{\tau}}\right]e^{-z^2}\,dz$$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f''(x+c\tau-2\sqrt{\tau}z)e^{-z^2}\,dz$

Since $\frac{\partial}{\partial z} f'(x+c\tau-2\sqrt{\tau}z) = f''(x+c\tau-2\sqrt{\tau}z)\cdot -2\sqrt{\tau}$

integration by parts gives

$$\frac{\partial^2 u}{\partial x^2} = \underbrace{-\frac{1}{2\sqrt{\tau}} f'(x+c\tau-2\sqrt{\tau}z)e^{-z^2}\Big]_{z=-\infty}^{z=\infty}}_{=0} + \frac{1}{2\sqrt{\tau}}\int_{-\infty}^{\infty} f'(x-c\tau-2\sqrt{\tau}z)e^{-z^2}\cdot -2z\,dz$$

$$= -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f'(x+c\tau-2\sqrt{\tau}z)\frac{z}{\sqrt{\tau}}e^{-z^2}\,dz$$

We conclude that $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}$

(c) $u(x,0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)e^{-z^2}\,dz = \frac{f(x)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz = f(x).$