Faculty of Bèta Sciences
Department Mathematics, Vrije Universiteit

Applied Analysis/Financial Mathematics
20-12-2017, 15:15–18:00

Instructions

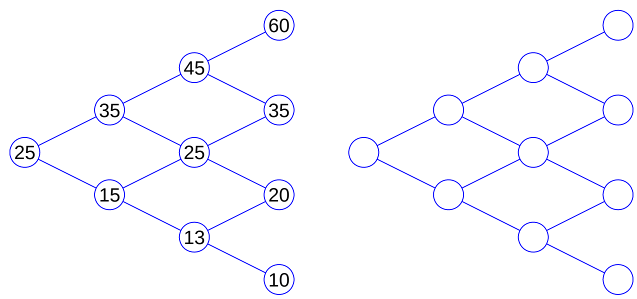
1. You are allowed to use a calculator during this exam.
2. Please write your name on the front page of the exam *before* you answer the questions. Write your name on every sheet of paper you hand in.
3. This exam consists of 5 questions
4. Answering instruction
   (a) Write your answers on the paper provided.
   (b) If you are asked to present an argument for your answer or to explain some issue, write your answers in correct sentences.
5. Maximal points per exercise are given in the following table.

question	a	b	c	d	total
1.	2	3	2	2	9
2.	3	3	3		9
3.	3	3	2	2	10
4.	4	4			8
5.	4	2	3		9
total					45

   Your grade for this exam is the number of points devided by 5 plus 1.

6. Your **mobile phone should be switched off** and should be put in your bag. Your bag needs to be closed. Any other electronic device, such as Ipads, laptops etc. should also be switched off and in your bag.

1. Given is the stock price in a binomial tree as follows:



So at time $t = 0$ the stock price S_0 is 25, etcetera. You may assume in this exercise that the interest rate $r = 0$.

a. Determine the martingale probabilities q for every fork of the tree.

b. Consider the cash or nothing option, paying out $B = 30$ at $t = 3$ when the stock price is above $E = 30$. Determine the price of the option at every node in the tree.

c. For all nodes at time level two, determine the replicating portfolio, i.e. determine the values of Δ and Π.

d. Explain why the structure of the replicating portfolio does not change for the top and bottom node at time level two when E changes from 30 to any value between 20 and 35.

2. Given is a portfolio consisting of a European put option and a European call option on the same stock S, with the same expiry date T, but with different exercise prices. The put option has exercise price E_p, the call option has exercise price E_c. So the value of the portfolio is $V(S,t) = P(S,t,T,E_p) + C(S,t,T,E_c)$.

a. Determine and sketch the pay-off function. Distinguish between the case $E_p < E_c$ and $E_p \geq E_c$.

b. In case $E_c < E_p$ show that $V(S,t) \geq (E_p - E_c)e^{-r(T-t)}$.

c. What are the appropriate boundary conditions for this option?

3. The European asset or nothing option $V(S,t)$ has a payoff equal to S when $S \geq E$ on the expiry date T, and a payoff of zero when $S < E$. Recall that this option has the value $V(S,t) = SN(d_1)$, where $d_1 = \frac{\ln \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$. As usual σ denotes the volatility of the share. (You do not need to show this, you may accept this as given.)

a. Compute $\Delta = \frac{\partial V}{\partial S}$.

b. Compute $\lim_{t \uparrow T} \Delta$. Distinguish three cases: $S < E$, $S > E$ and $S = E$. (It helps to sketch the payoff function.)

c. Explain why the result from part **b** is to be expected when one considers the payoff function.

d. Would it be possible in practice to maintain a replicating portfolio for this option?

4. Consider the partial differential equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x} - v,$$

with initial condition $v(x,0) = v_0(x)$.

a. Let $u(x,\tau) = e^{-(\alpha x + \beta \tau)} v(x,\tau)$. Determine the values of α and β for which u satisfies the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

b. Prove that the solution $v(x,\tau)$ is given by

$$v(x,\tau) = \frac{e^{-x-2\tau}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v_0(s) e^s e^{-(x-s)^2/4\tau} ds.$$

5. Consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial u}{\partial x}, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x,0) = f(x)$.

In this exercise we shall consider a numerical method to solve this equation. As usual, consider a grid with stepsize δx in the x direction, $\delta\tau$ in the τ direction, and denote $u(n\delta x, m\delta\tau)$ by u_n^m.

a. Use the forward difference for the τ derivative, the symmetric central difference for the second order derivative in x and the central difference for the first derivative in x. Give a formula expressing u_n^{m+1} in terms of $u_{n-1}^m, u_n^m, u_{n+1}^m$. Use $\alpha = \frac{\delta\tau}{(\delta x)^2}$ and $\beta = \frac{\delta\tau}{2\delta x}$.

b. How should we take u_n^0?

c. Discuss stability of the method in terms of α and β.