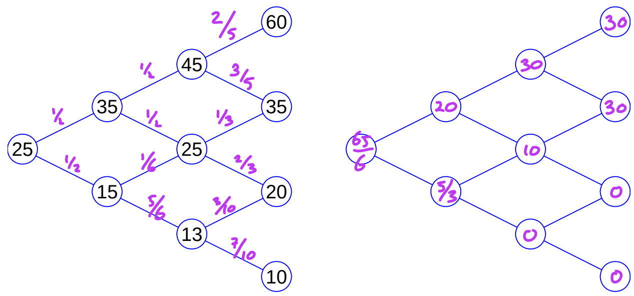
1. Given is the stock price in a binomial tree as follows:



So at time $t = 0$ the stock price S_0 is 25, etcetera. You may assume in this exercise that the interest rate $r = 0$.

a. Determine the martingale probabilities q for every fork of the tree.

b. Consider the cash or nothing option, paying out $B = 30$ at $t = 3$ when the stock price is above $E = 30$. Determine the price of the option at every node in the tree.

c. For all nodes at time level two, determine the replicating portfolio, i.e. determine the values of Δ and Π.

d. Explain why the structure of the replicating portfolio does not change for the top and bottom node at time level two when E changes from 30 to any value between 20 and 35.

(a) for $r=0$ we have $q = \dfrac{S_0 - S_d}{S_u - S_d}$ The values are indicated in the diagram above

(b) $V_0 = q V_u + (1-q) S_d$ at every node, starting at $t=3$,
where the payoff is 30 if $S > 30$ and 0 if $S < 30$.
The values are given in the diagram above
leading to the option price $V = \frac{65}{6}$ at $t=0$.

(c) $\Delta = \dfrac{V_u - V_d}{S_u - S_d}$ and $\Pi = V - \Delta S$ At $t=2$ this gives $\Delta = 0$ $\Pi = 30$

$\Delta = 2$ $\Pi = -40$

$\Delta = 0$ $\Pi = 0$ for the replicating portfolio.

(d) Since the pay-off doesn't change, neither do Δ and Π. (for any node in fact)

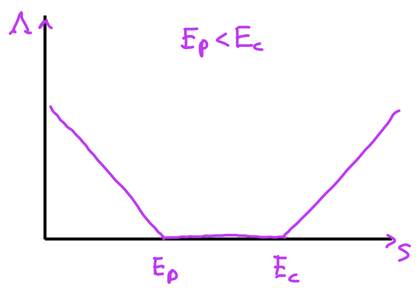
2. Given is a portfolio consisting of a European put option and a European call option on the same stock S, with the same expiry date T, but with different exercise prices. The put option has exercise price E_p, the call option has exercise price E_c. So the value of the portfolio is $V(S,t) = P(S,t,T,E_p) + C(S,t,T,E_c)$.

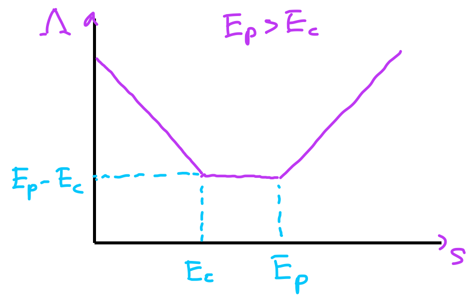
a. Determine and sketch the pay-off function. Distinguish between the case $E_p < E_c$ and $E_p \geq E_c$.

b. In case $E_c < E_p$ show that $V(S,t) \geq (E_p - E_c)e^{-r(T-t)}$.

c. What are the appropriate boundary conditions for this option?

ⓐ $\Lambda(S) = (E_p - S)_+ + (S - E_c)_+$





ⓑ Since $V(S,T) \geq E_p - E_c$ without risk
the value of the option $V(S,t)$ is at least the discounted value $e^{-r(T-t)}(E_p - E_c)$

ⓒ By boundary conditions we main $V(0,t)$ at $S=0$ and the limit behavioer of $V(S,t)$ as $S \to \infty$.
We have $V(0,t) = e^{-r(T-t)} V(0,T) = e^{-r(T-t)} E_p$
and $\lim_{S \to \infty} \frac{V(S,t)}{S} = 1$

3. The European asset or nothing option $V(S,t)$ has a payoff equal to S when $S \geq E$ on the expiry date T, and a payoff of zero when $S < E$. Recall that this option has the value $V(S,t) = SN(d_1)$, where $d_1 = \frac{\ln \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$. As usual σ denotes the volatility of the share. (You do not need to show this, you may accept this as given.)

a. Compute $\Delta = \frac{\partial V}{\partial S}$.

b. Compute $\lim_{t \uparrow T} \Delta$. Distinguish three cases: $S < E$, $S > E$ and $S = E$. (It helps to sketch the payoff function.)

c. Explain why the result from part **b** is to be expected when one considers the payoff function.

d. Would it be possible in practice to maintain a replicating portfolio for this option?

(a) $\Delta = \frac{\partial V}{\partial S} = N(d_1) + S\,N'(d_1)\frac{\partial d_1}{\partial S}$ $\qquad N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-s^2/2}\,ds \Rightarrow N'(d) = e^{-d^2/2}$

$\frac{\partial d_1}{dS} = \frac{1}{\sigma\sqrt{T-t}}\cdot\frac{1}{S}$

Hence $\Delta = N(d_1) + \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}}$

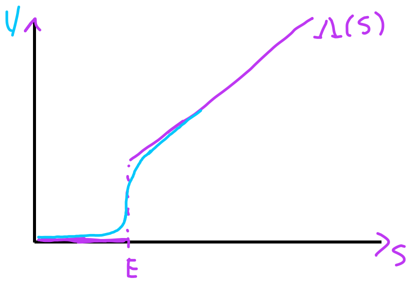
(b) $\lim_{t\uparrow T} d_1 = \lim_{t\uparrow T} \frac{1}{\sigma\sqrt{T-t}}\cdot\log\frac{S}{E} + \frac{(r+\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t} = \begin{cases} -\infty & \text{if } S>E \\ 0 & \text{if } S=E \\ \infty & \text{if } S>E \end{cases}$

Case $S<E$: $\lim_{t\uparrow T}\Delta = \lim_{t\uparrow T} N(d_1) + \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} = \underbrace{\lim_{d\to-\infty} N(d)}_{=0} + \lim_{t\uparrow T}\frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} = 0 + 0 = 0$

since $e^{-d_1^2/2} \to 0$ much more quickly than $\sqrt{T-t} \to 0$.

Case $S=E$: $\lim_{t\uparrow T}\Delta = \lim_{t\uparrow T} N(d_1) + \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} = N(0) + \infty = \infty$

Case $S>E$: $\lim_{t\uparrow T}\Delta = \lim_{t\uparrow T} N(d_1) + \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} = \underbrace{\lim_{d\to\infty} N(d)}_{=1} + \lim_{t\uparrow T}\frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} = 1 + 0 = 1.$



(c) $\lim_{t\uparrow T} V(S,t) = V(S,T) = \Lambda(S)$ and $\frac{d\Lambda}{dS}(S) = \begin{cases} 0 & \text{if } S<E \\ \text{undefined} & S=E \\ 1 & S>E. \end{cases}$ since $\Lambda(S)$ has a jump "up", hence "$\frac{d\Lambda}{dS}(E) = \infty$".

(d) No. As Δ becomes enormous if the stock price S is close to E when t approaches T this would mean buying the entire company.

4. Consider the partial differential equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x} - v,$$

with initial condition $v(x,0) = v_0(x)$.

a. Let $u(x,\tau) = e^{-(\alpha x+\beta\tau)}v(x,\tau)$. Determine the values of α and β for which u satisfies the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

b. Prove that the solution $v(x,\tau)$ is given by

$$v(x,\tau) = \frac{e^{-x-2\tau}}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} v_0(s)e^{s}e^{-(x-s)^2/4\tau}\,ds.$$

(a)

$$\frac{\partial u}{\partial \tau} = -\beta e^{-(\alpha x+\beta\tau)}v + e^{-(\alpha x+\beta\tau)}\frac{\partial v}{\partial \tau} = -\beta e^{-(\alpha x+\beta\tau)}v + e^{-(\alpha x+\beta\tau)}\left[\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x} - v\right]$$

$$\frac{\partial u}{\partial x} = -\alpha e^{-(\alpha x+\beta\tau)}v + e^{-(\alpha x+\beta\tau)}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \alpha^2 e^{-(\alpha x+\beta\tau)}v - 2\alpha e^{-(\alpha x+\beta\tau)}\frac{\partial v}{\partial x} + e^{-(\alpha x+\beta\tau)}\frac{\partial^2 v}{\partial x^2}.$$

We have $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ if

$$-\beta e^{-(\alpha x+\beta\tau)}v + e^{-(\alpha x+\beta\tau)}\left[\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x} - v\right] = \alpha^2 e^{-(\alpha x+\beta\tau)}v - 2\alpha e^{-(\alpha x+\beta\tau)}\frac{\partial v}{\partial x} + e^{-(\alpha x+\beta\tau)}\frac{\partial^2 v}{\partial x^2}.$$

which is true if

$$\begin{cases} -\beta - 1 = \alpha^2 \\ 2 = -2\alpha \end{cases} \quad \text{hence if} \quad \begin{cases} \alpha = -1 \\ \beta = -2 \end{cases}$$

(b) Since $u(x,0) = e^{-\alpha x}v(x,0) = e^{x}v(x,0) = e^{x}v_0(x)$

and $u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u(s,0)e^{-\frac{(x-s)^2}{4\tau}}ds$

we find

$$v(x,\tau) = e^{+(\alpha x+\beta\tau)}u(x,\tau) = \frac{e^{-x-2\tau}}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{s}v_0(s)e^{-\frac{(x-s)^2}{4\tau}}ds.$$

5. Consider the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + 3\frac{\partial u}{\partial x}, \qquad \tau > 0, x \in \mathbb{R},$$

with initial condition $u(x,0) = f(x)$.

In this exercise we shall consider a numerical method to solve this equation. As usual, consider a grid with stepsize δx in the x direction, $\delta\tau$ in the τ direction, and denote $u(n\delta x, m\delta\tau)$ by u_n^m.

a. Use the forward difference for the τ derivative, the symmetric central difference for the second order derivative in x and the central difference for the first derivative in x. Give a formula expressing u_n^{m+1} in terms of $u_{n-1}^m, u_n^m, u_{n+1}^m$. Use $\alpha = \frac{\delta\tau}{(\delta x)^2}$ and $\beta = \frac{\delta\tau}{2\delta x}$.

b. How should we take u_n^0?

c. Discuss stability of the method in terms of α and β.

a) $\frac{\partial u}{\partial \tau}(n\delta x, m\delta\tau) \approx \frac{u_n^{m+1} - u_n^m}{\delta\tau}$ $\quad$ $\frac{\partial u}{\partial x}(n\delta x, m\delta\tau) \approx \frac{u_{n+1}^m - u_{n-1}^m}{2\delta x}$

$\frac{\partial^2 u}{\partial x^2}(n\delta x, m\delta\tau) \approx \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{(\delta x)^2}$

$\Longrightarrow \frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{(\delta x)^2} + 3\frac{u_{n+1}^m - u_{n-1}^m}{2\delta x}$

$\Longrightarrow u_n^{m+1} - u_n^m = \frac{\delta\tau}{(\delta x)^2}\left(u_{n-1}^m - 2u_n^m + u_{n+1}^m\right) + 3\cdot\frac{\delta\tau}{2\delta x}\left(u_{n+1}^m - u_{n-1}^m\right)$

$= \alpha\left(u_{n-1}^m - 2u_n^m + u_{n+1}^m\right) + 3\beta\left(u_{n+1}^m - u_{n-1}^m\right)$

$\Longrightarrow u_n^{m+1} = (\alpha - 3\beta)u_{n-1}^m + (1-2\alpha)u_n^m + (\alpha+3\beta)u_{n+1}^m$

b) $u_n^0 = u(n\delta x, 0) = f(n\delta x)$.

c) $\alpha = \frac{\delta\tau}{(\delta x)^2}$ and $\beta = \frac{\delta\tau}{2\delta x}$ we have $\beta = 2\alpha\,\delta x$

Since δx is tiny β is much smaller than α, hence β is negligable.

The stability properties are thus the same as for the explicit scheme for $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ discussed in the course, hence $\alpha \le \frac{1}{2}$ is the criterion to have a stable numerical scheme.