Faculty of Science	Midterm exam Analysis II
Department of Mathematics	29-03-2022
Vrije Universiteit Amsterdam	12:15-14:30

The use of a calculator, a book, or lecture notes is not permitted.
Do not just give answers, but give calculations and explain your steps.

1. Solve the initial value problem

$$\begin{cases} y'(x) + e^{2y(x)} \cdot (2x^3 + x) = 0, \\ y(0) = 0. \end{cases}$$

2. Find the general solution of

$$4y''(x) - 4y'(x) + 5y(x) = 5x + 1.$$

3. Prove in two different ways that the series $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n^2 - 2n}$ is convergent:

   a) using the Limit Comparison Test,

   b) using the Integral Comparison Test.

4. Determine if the following series are convergent or divergent. If the series is convergent explain if it is conditionally convergent or absolute convergent.

$$\text{a) } \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1 + e^{-n}}{\sqrt{n}}, \qquad \text{b) } \sum_{n=1}^{\infty} \frac{(n+1)!}{2^n \cdot n^2}.$$

5. Consider the sequence of functions $\{f_n\}$ defined by $f_n(x) = \dfrac{nx^2 + \cos x}{n+1}$.

   a) Prove that $\{f_n\}$ converges pointwise for all $x \in \mathbb{R}$.

   b) Determine if $\{f_n\}$ converges uniformly on $[-1, +1]$.

   c) Compute $\displaystyle\lim_{n\to\infty} \int_{-1}^{+1} f_n(x)\, dx$.

Please turn over

6. Consider the series of functions $\sum_{n=1}^{\infty} \frac{\sin(nx+1)}{n^2+x^2}$.

   a) Prove that this series converges normally (= absolutely uniformly) to some function F on $[1, \infty)$.

   b) Is the sum function F continuous on $[1, \infty)$? Explain your answer.

7. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^{n+2}}{n \cdot 4^n}$.

   a) Determine its radius of convergence.

   b) Determine its interval of convergence.

8. Consider the function $f(x) = \cos(x^2 - 2x + 1)$.

   a) Find the power series representation of $f(x)$ centered at $x = 1$.

   b) Use your result to express $\int_1^2 f(x)\,dx$ as the sum of an alternating series of real numbers.

Scores:

1 : 3	2 : 3	3 : a) 2 b) 3	4 : a) 3 b) 2	5 : a) 2 b) 2 c) 2	6 : a) 3 b) 1	7 : a) 2 b) 3	8 : a) 3 b) 2
3	3	5	5	6	4	5	5

$$\text{Grade} = \frac{\#\text{ points}}{4} + 1$$