Faculty of Science	Midterm exam Analysis II
Department of Mathematics	29-03-2022
Vrije Universiteit Amsterdam	12:15-14:30

The use of a calculator, a book, or lecture notes is <u>not</u> permitted.
Do not just give answers, but give calculations and explain your steps.

1. Using separation of variables we get $-e^{-2y}\,dy = (2x^3 + x)\,dx$. Integration on both sides and dividing by $\frac{1}{2}$ we get $e^{-2y} = x^4 + x^2 + C$ and hence $y = -\frac{1}{2}\ln(x^4 + x^2 + C)$ with $C = 1$ for $y(0) = 0$.

2. First we solve the characteristic equation $4r^2 - 4r + 5 = 0$ with discriminant $D = -8^2$ in order to get $r_{1,2} = \frac{1}{2} \pm i$, so that the general solution of the corresponding homogeneous equation is given by $y(x) = e^{\frac{1}{2}x}(A\cos x + B\sin x)$ for $A, B \in \mathbb{R}$. In order to find a particular solution we insert the ansatz $y_p(x) = ax + b$ in order to get $5ax + (5b - 4a) = 5x + 1$ and hence $a = b = 1$. The general solution is hence $y(x) = e^{\frac{1}{2}x}(A\cos x + B\sin x) + x + 1$ with $A, B \in \mathbb{R}$.

3. a) $\lim_{n\to\infty}\left(\frac{2}{n^2 - 2n}/\frac{1}{n^2}\right) = 2$ and $\sum_{n=3}^{\infty}\frac{1}{n^2}$ converges.

   b) Since $\frac{2}{x(x-2)} = \frac{1}{x-2} - \frac{1}{x}$ we have $\int_3^{\infty}\frac{2}{x(x-2)}\,dx = \lim_{b\to\infty}(\ln(x-2) - \ln x)\Big|_3^b$ with $\lim_{b\to\infty}(\ln(b-2) - \ln b) = \lim_{b\to\infty}\ln\left(1 - \frac{2}{b}\right) = 0$.

4. a) Since $1 + e^{-n}$ decreases and $\sqrt{n}$ increases with n, $f(n) = \frac{1 + e^{-n}}{\sqrt{n}}$ is decreasing. Together with $f(n) > 0$ and $\lim_{n\to\infty} f(n) = 0$, we get convergence of the series using the Leibniz Alternating Series Test. Since $\frac{1 + e^{-n}}{\sqrt{n}} > \frac{1}{\sqrt{n}}$ and $\sum_{n\to\infty}\frac{1}{\sqrt{n}}$ diverges, we have that $\sum_{n=1}^{\infty}\frac{1 + e^{-n}}{\sqrt{n}}$ diverges by the Comparison Test. Hence the series only converges conditionally.

   b) For $a_n := \frac{(n+1)!}{2^n \cdot n^2}$ we have $\frac{a_{n+1}}{a_n} = \frac{n+2}{2}\cdot\frac{n^2}{(n+1)^2}$ and hence $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \infty$, so the series diverges using the Limit Ratio Test.

5. a) $\lim_{n\to\infty}\frac{nx^2 + \cos x}{n+1} = \lim_{n\to\infty}\frac{x^2 + \frac{1}{n}\cos x}{1 + \frac{1}{n}} = x^2 = f(x)$ for all $x \in \mathbb{R}$.

   b) $|f_n(x) - f(x)| = \frac{|\cos x - x^2|}{n+1} \le \frac{2}{n+1}$ for $|x| \le 1$. Since $\lim_{n\to\infty}\frac{2}{n+1} = 0$, we have uniform convergence.

   c) $\lim_{n\to\infty}\int_{-1}^{+1} f_n(x)\,dx = \int_{-1}^{+1}\left(\lim_{n\to\infty} f_n(x)\right)\,dx = \int_{-1}^{+1} x^2\,dx = \frac{2}{3}$.

6. a) Since $\left|\frac{\sin(nx+1)}{n^2+x^2}\right| < \frac{1}{n^2}$ for $x \in [1,\infty)$, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, normal (= absolutely uniform) convergence of the series follows from Weierstrass M-test.

b) Since normal convergence implies uniform convergence, the limit function is continuous.

7. a) For $a_n = \frac{1}{(n-2)\cdot 4^{n-2}}$, we have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}\cdot\lim_{n\to\infty}\frac{n-1}{n-2} = \frac{1}{4}$ and hence the radius of convergence is $R = 4$.

b) For $x = 7$, we have that $\sum_{n=1}^{\infty}\frac{(7-3)^{n+2}}{n\cdot 4^n} = 16\cdot\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (as p-series with $p = 1$), while for $x = -1$ we have that $\sum_{n=1}^{\infty}\frac{(-1-3)^{n+2}}{n\cdot 4^n} = 16\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges (by the Leibniz alternating series test). By (a), we have convergence if $|x-3| < 4$ and divergence if $|x-3| > 4$, so that the interval of convergence is $I = [-1, 7)$.

8. a) $f(x) = \cos((x-1)^2) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{((x-1)^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{(x-1)^{4n}}{(2n)!}$.

b) $\int_1^2 f(x)\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot\int_1^2 (x-1)^{4n}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\cdot(4n+1)}$.

Scores:

1	2	3	4	5	6	7	8
1 : 3	2 : 3	3 : a) 2	4 : a) 3	5 : a) 2	6 : a) 3	7 : a) 2	8 : a) 3
		b) 3	b) 2	b) 2	b) 1	b) 3	b) 2
				c) 2			
3	3	5	5	6	4	5	5

$$\text{Grade} = \frac{\#\text{ points}}{4} + 1$$