Answers 2nd partial exam Analysis II (26-05-2021)

Question 1. [4 points] Prove or disprove the following statement:

"The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : 3 < x_1 < 7, 2 \leq x_2 \leq 5\}$ is an open set."

Answer:
The statement is false.

The point $(4, 2) \in A$. However, for every $r > 0$, $(4, 2 - \frac{r}{2}) \in B((4, 2), r)$, but $(4, 2 - \frac{r}{2}) \notin A$. Therefore, we have found a point in A for which there is no $r > 0$ with $B((4, 2), r) \subseteq A$. This shows that A is not open.

Question 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^4 - x_1 x_2^2 + x_2^4}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0, 0), \\ 0 & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

(a) [4 points] Is f continuous at $(0, 0)$?
Answer:
To show that f is continuous at $(0, 0)$, let $\epsilon > 0$ and $\delta = \min\{1, \frac{\epsilon}{3}\}$. Let $(x_1, x_2) \in \mathbb{R}^2$ with $0 < ||(x_1, x_2)|| < \delta$. Then,

$$\begin{aligned} \left| \frac{x_1^4 - x_1 x_2^2 + x_2^4}{x_1^2 + x_2^2} - 0 \right| &\underset{\text{Triangle ineq.}}{\leq} \frac{x_1^4 + |x_1| x_2^2 + x_2^4}{x_1^2 + x_2^2} \\ &\underset{x_1^2, x_2^2 \geq 0}{\leq} \frac{x_1^2(x_1^2 + x_2^2) + |x_1|(x_1^2 + x_2^2) + x_2^2(x_1^2 + x_2^2)}{x_1^2 + x_2^2} \\ &= x_1^2 + |x_1| + x_2^2 \underset{||\boldsymbol{x}|| < \delta}{<} \delta^2 + \delta + \delta^2 \underset{\delta \leq 1}{\leq} 3\delta \leq \epsilon. \end{aligned}$$

This shows that f is continuous at $(0, 0)$.

(b) [1 point] Calculate $\frac{\partial f}{\partial x_1}(0, 0)$ and $\frac{\partial f}{\partial x_2}(0, 0)$.
Answer:

$$\frac{\partial f}{\partial x_1}(0, 0) = \lim_{h_1 \to 0} \frac{f(h_1, 0) - f(0, 0)}{h_1} = \lim_{h_1 \to 0} \frac{\frac{h_1^4 - 0 + 0}{h_1^2 + 0} - 0}{h_1} = \lim_{h_1 \to 0} h_1 = 0$$

and

$$\frac{\partial f}{\partial x_2}(0, 0) = \lim_{h_2 \to 0} \frac{f(0, h_2) - f(0, 0)}{h_2} = \lim_{h_2 \to 0} \frac{\frac{0 - 0 + h_2^4}{0 + h_2^2} - 0}{h_2} = \lim_{h_2 \to 0} h_2 = 0.$$

(c) [4 points] Is f (totally) differentiable at $(0,0)$?
Answer:
To show that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2)-f(0,0)-f_{x_1}(0,0)h_1-f_{x_2}(0,0)h_2}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^4-h_1h_2^2+h_2^4}{(h_1^2+h_2^2)^{\frac{3}{2}}} \neq 0,$$

take $h_1 = r\cos\phi$ and $h_2 = r\sin\phi$, $\phi \in \mathbb{R}$. Then,

$$\begin{aligned}\lim_{h_1\to 0} \frac{h_1^4-h_1h_2^2+h_2^4}{(h_1^2+h_2^2)^{\frac{3}{2}}} &= \lim_{r\to 0} \frac{r^4(\cos^4\phi+\sin^4\phi)-r^3\cos\phi\sin^2\phi}{(r^2)^{\frac{3}{2}}} \\ &= \lim_{r\to 0}(r(\cos^4\phi+\sin^4\phi)-\cos\phi\sin^2\phi) = -\cos\phi\sin^2\phi\end{aligned}$$

which depends on the value of ϕ. Therefore, $\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2)-f(0,0)-f_{x_1}(0,0)h_1-f_{x_2}(0,0)h_2}{\sqrt{h_1^2+h_2^2}}$ does not exists and f is not differentiable at $(0,0)$.

Question 3. Consider the equation

$$xyz + 2z\cos y + y\sin z = 2\pi.$$

(a) [2 points] Can the equation be solved for z in terms of x and y in a neighborhood of $(\frac{4}{\pi}, \pi, \frac{\pi}{2})$?
Answer:
Let $F(x,y,z) = xyz + 2z\cos y + y\sin z - 2\pi$. Then, F has continuous first order partial derivatives and

(i) $F(\frac{4}{\pi}, \pi, \frac{\pi}{2}) = 2\pi + \pi\cos(\pi) + \pi\sin(\frac{\pi}{2}) - 2\pi = 2\pi - \pi + \pi - 2\pi = 0$.

(ii) $F_z(\frac{4}{\pi}, \pi, \frac{\pi}{2}) = xy + 2\cos y + y\cos z_{|(\frac{4}{\pi},\pi,\frac{\pi}{2})} = 4 - 2 + 0 = 2 \neq 0$.

Then, we can apply the Implicit Function Theorem and it turns out that the equation $xyz+2z\cos y+y\sin z = 2\pi$ can be solved uniquely for z in terms of x, y in a neighborhood of $(\frac{4}{\pi}, \pi, \frac{\pi}{2})$, i.e., $z = g(x,y)$.

(b) [3 points] Determine $\frac{\partial z}{\partial x}$ in a neighborhood of $(\frac{4}{\pi}, \pi, \frac{\pi}{2})$.
Answer:
By (a), we know that $G(x,y) = F(x,y,g(x,y)) = 0$ for all (x,y) in a neighborhood of $(\frac{4}{\pi}, \pi)$. Since $G(x,y)$ is constant, we have $G_x(x,y) = 0$ for all (x,y) in our neighborhood. Then, using the Chain rule, we have

$$\begin{aligned}G_x(x,y) &= F_x(x,y,g(x,y))\frac{\partial x}{\partial x} + F_y(x,y,g(x,y))\frac{\partial y}{\partial x} + F_z(x,y,g(x,y))\frac{\partial g(x,y)}{\partial x} \\ &= yg(x,y)\cdot 1 + (xg(x,y) - 2g(x,y)\sin y + \sin(g(x,y)))\cdot 0 \\ &\quad + (xy + 2\cos y + y\cos(g(x,y)))\frac{\partial g(x,y)}{\partial x} = 0\end{aligned}$$

Therefore,

$$\frac{\partial z}{\partial x} = \frac{\partial g(x,y)}{\partial x} = \frac{-yg(x,y)}{xy + 2\cos y + y\cos(g(x,y))}.$$

Question 4. [7 points] Find, if it exists, the minimum of $\frac{1}{x^2+y^2}$ subject to the constraint $5x^2 + 5y^2 + 6xy = 4$ using Lagrange multipliers.

Answer:

Since our restriction describes an ellipse centered at $(0, 0)$, we know that the function attains a maximum and a minimum under our constraint by the Extreme Value Theorem.

Besides, since $g(z) = 1/z$ is a decreasing function on $(0, \infty)$, solving

$$\begin{array}{cc} \min & \frac{1}{x^2+y^2} \\ \text{s.t.} & \\ & 5x^2 + 5y^2 + 6xy - 4 = 0. \end{array}$$

is equivalent to solving the problem

$$\begin{array}{cc} \max & x^2 + y^2 \\ \text{s.t.} & \\ & 5x^2 + 5y^2 + 6xy - 4 = 0. \end{array}$$

The Lagrange function is $\mathcal{L}(x_1, x_2, \lambda) = x^2 + y^2 + \lambda(5x^2 + 5y^2 + 6xy - 4)$. The maximum is obtained in the critical values of the Lagrange function. We get the system of equations

$$\left.\begin{array}{l} 2x + \lambda(10x + 6y) = 0 \\ 2y + \lambda(6x + 10y) = 0 \\ 5x^2 + 5y^2 + 6xy = 4 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 2x(6x + 10y) = -\lambda(10x + 6y)(6x + 10y) \\ 2y(10x + 6y) = -\lambda(6x + 10y)(10x + 6y) \\ 5x^2 + 5y^2 + 6xy = 4 \end{array}\right|$$

$$\Leftrightarrow \left.\begin{array}{l} 2x + \lambda(10x + 6y) = 0 \\ 2y(10x + 6y) = 2x(6x + 10y) \Leftrightarrow y^2 = x^2 \\ 5x^2 + 5y^2 + 6xy = 4 \end{array}\right|$$

For $y = x$, substituting in $5x^2 + 5y^2 + 6xy = 4$, we have

$$10x^2 + 6x^2 = 4 \Leftrightarrow x = \pm\frac{1}{2}$$

and we get two points: $(\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$.

For $y = -x$, , substituting in $5x^2 + 5y^2 + 6xy = 4$, we have

$$10x^2 - 6x^2 = 4 \Leftrightarrow x = \pm 1$$

and we get two points: $(1, -1)$ and $(-1, 1)$.

Then, our objective function $f(x_1, x_2) = \frac{1}{x^2+y^2}$ attains the minimum in (some of) the four critical points: $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(1, -1)$ and $(-1, 1)$. Since $f(\frac{1}{2}, \frac{1}{2}) = f(-\frac{1}{2}, -\frac{1}{2}) = 2$ and $f(1, -1) = f(-1, 1) = \frac{1}{2}$, the minimum is $\frac{1}{2}$ and attained at $(1, -1)$ and $(-1, 1)$.

Question 5. [4 points] Calculate the double integral

$$\iint_G ye^{x^2+3} dA$$

where G is the triangle with corner points $(0,1)$, $(1,0)$, $(1,2)$.
Answer:

$$\iint_G ye^{x^2+3}dA = \int_0^1 \int_{1-x}^{1+x} ye^{x^2+3}dydx = \int_0^1 \left|\frac{y^2}{2}\right|_{1-x}^{1+x} e^{x^2+3}dx = \int_0^1 2xe^{x^2+3}dx$$
$$= \int_0^1 (x^2+3)'e^{x^2+3}dx = \left|e^{x^2+3}\right|_0^1 = e^4 - e^3 = e^3(e-1).$$

Question 6. [7 points] Calculate the double integral

$$\iint_S \frac{x_1x_2}{(x_1^2+x_2^2)^3}dA$$

where $S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1, x_1 \geq 0, 0 \leq x_2 \leq x_1\}$.
Answer:
Since S is not bounded, we have an improper integral of the first type. Using change of variable to polar coordinates: $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $dA = rdrd\phi$, and $S_{(r,\phi),m} = \{(r,\phi) \in \mathbb{R}^2 | 1 \leq r \leq m, 0 \leq \phi \leq \frac{\pi}{4}\}$

$$\iint_S \frac{x_1x_2}{(x_1^2+x_2^2)^3}dA = \lim_{m\to\infty} \iint_{S_m} \frac{x_1x_2}{(x_1^2+x_2^2)^3}dA = \lim_{m\to\infty} \int_0^{\frac{\pi}{4}} \int_1^m \frac{r^2\cos(\phi)\sin(\phi)}{(r^2)^3}rdrd\phi$$
$$= \lim_{m\to\infty} \int_0^{\frac{\pi}{4}} \int_1^m \frac{\cos(\phi)\sin(\phi)}{r^3}drd\phi = \lim_{m\to\infty} \int_0^{\frac{\pi}{4}} \left|\frac{-1}{2r^2}\right|_1^m (\sin(\phi))'\sin(\phi)d\phi$$
$$= \lim_{m\to\infty} \left(\frac{1}{2} - \frac{1}{2m^2}\right) \left|\frac{1}{2}\sin^2(\phi)\right|_0^{\frac{\pi}{4}} = \lim_{m\to\infty} \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2m^2}\right) = \frac{1}{8}.$$