1.
$$\begin{cases} y'(x) + \frac{3}{x}y = \frac{2}{x^2} + e^{x^4}, \\ y(1) = 1. \end{cases}$$

The ODE is the first order Linear type, which can be solved by using the integrating factor:

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\int \frac{dx}{x}} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3.$$

Hence,

$$\left(y' + \frac{3}{x}y\right)x^3 = y'x^3 + \frac{3}{x}yx^3 = y'x^3 + 3yx^2 = \left(yx^3\right)'.$$

$$\left(yx^3\right)' = x^3(\frac{2}{x^2} + e^{x^4}) = 2x + x^3e^{x^4}.$$

Then, by taking the integral of the lats equation we obtain the general solution:

$$\left(yx^3\right) = x^2 + \frac{e^{x^4}}{4} + C,$$

$$y(x) = \frac{1}{x} + \frac{e^{x^4}}{4x^3} + \frac{C}{x^3}.$$

Substitution of the initial value to the general solution we get

$$1 = y(1) = 1 + \frac{e}{4} + C.$$

So the solution of this initial value problem is

$$y(x) = \frac{1}{x} + \frac{e^{x^4}}{4x^3} + \frac{-e}{4x^3}.$$

2. First solve the homogeneous equation. Substitution of $y(x) = e^{rx}$ gives the auxiliary equation $r^2 - 6r + 13 = 0$, with the solutions $r = 3 \pm 2i$. Hence, the general solution of the homogeneous equation is

$$y(x) = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x), C_1, C_2 \in \mathbb{R}.$$

A particular solution will be of the form $y_p(x) = Ax + B + Ce^x$. Then we have $y_p'(x) = A + Ce^x$ and $y_p''(x) = Ce^x$. Substitution in the nonhomogeneous equation yields

$$8C = 1,\ -6A + 13B = 0,\ 13A = 1.$$

So, $\left\{A = \frac{1}{13}, B = \frac{6}{169}, C = \frac{1}{8}\right\}$. The general solution for this inhomogeneous differential equation is therefore:

$$y(x) = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + \frac{1}{13}x + \frac{6}{169} + \frac{1}{8}e^x, C_1, C_2 \in \mathbb{R}.$$

3. a) First consider

$$\sum_{n=1}^{\infty} |(-1)^n \frac{(n^2+4n+2)}{2^n e^n}| = \sum_{n=1}^{\infty} \frac{(n^2+4n+2)}{2^n e^n}$$

Using the ratio test one obtains:

$$\lim_{n\to\infty} \frac{((n+1)^2+4n+6)}{2^{n+1}e^{n+1}} : \frac{(n^2+4n+2)}{2^n e^n} = \frac{1}{2e} < 1.$$

Hence, the series converges absolutely.

b)

$$\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}.$$

First consider,

$$\sum_{n=3}^{\infty} \frac{\ln(n)}{n}.$$

For all $n \geq 3$ the following holds true:

$$\frac{\ln(n)}{n} \geq \frac{1}{n}.$$

Since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, then $\sum\limits_{n=3}^{\infty} \frac{\ln(n)}{n}$ also diverges. So there is no absolute convergence. Now apply the alternating series test: (i) the series is alternating, (ii) the general term tends to 0 and (iii) the sequence $\frac{\ln(n)}{n}$ is decreasing for all $n \geq 3$. For the third item define $g(x) = \frac{\ln(x)}{x}$ so $g'(x) = \frac{1-\ln x}{x^2} \leq 0$. (Note that $\ln(x) \geq 1$ on the interval $[e, \infty)$).
So the series is convergent, but not absolutely convergent, therefore it is conditionally convergent.

4. a) Th pointwise limit is $f(x) = x$. Since,

$$\lim_{n\to\infty} \left(\frac{1}{n} x \sin(x) + x\right) = x = f(x).$$

Now, we will prove uniform convergence. Let $\varepsilon > 0$ be given. Then, for all $n \geq n^*$ we find that for each $x \in [a, b]$:

$$|f_n(x) - f(x)| = \left|\frac{1}{n} x \sin(x) + x - x\right| = \left|\frac{1}{n} x \sin(x)\right| \leq \frac{M}{n} < \varepsilon,$$

where $M = \max\{|a|, |b|\}$, Hence, for all $n \geq n^* > \frac{M}{\varepsilon}$ the sequence converges uniformly on the interval $[a, b]$.

b) The point wise limit is $f(x) = x$, but the sequence is not uniformly convergent on $\mathbb{R}$. For any $n \in \mathbb{N}$, we can always find $x \in \mathbb{R}$ large enough to guarantee that $\frac{1}{n}|x| \sin(x)| > 1$ by taking x, such that $|x|| \sin(x)| > n$. Thus for $\varepsilon = 1$ and any $n \in \mathbb{N}$ there exists $x \in \mathbb{R}$ such that

$$|f_n(x_n) - f(x_n)| \geq 1,$$

so the sequence $\{f_n\}$ does not converge uniformly on $\mathbb{R}$. For instance, set $x_n = 2n\pi + \frac{\pi}{2}$ then one has the following

$$|f_n(x_n) - f(x_n)| = \left|2\pi + \frac{\pi}{2n}\right| \geq 1, \text{ for every } n \in \mathbb{N}.$$

5.
$$\sum_{n=0}^{\infty} \frac{x}{(n+x^2)^2}$$

Define $f_n(x) = \frac{x}{(n+x^2)^2}$ for all $n \geq 0$. Then, $f_n'(x) = \frac{-3\,x^2+n}{(x^2+n)^3}$. So $f_n'(x) = 0$ for $x = \pm\frac{\sqrt{3n}}{3}$. Since $f_n''(\frac{\sqrt{3n}}{3}) = -\frac{27\sqrt{3}}{32}n^{-\frac{5}{2}}$ then $f_n(x)$ has its maximum at $x = \sqrt{\frac{n}{3}}$. Therefore,

$$|f_n(x)| \leq M_n = \frac{3\sqrt{3}}{16}n^{-\frac{3}{2}}.$$

Now, we apply the Weierstrass M-test. Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{3\sqrt{3}}{16}n^{-\frac{3}{2}}$ is p-series with $p > 1$, then the series converges. Weierstrass M-test concludes that $\sum\limits_{n=0}^{\infty} \frac{x}{(n+x^2)^2}$ is uniform convergent on $\mathbb{R}$.

6. a) Set $a_n = \dfrac{(3x+1)^{4n}}{n\ 16^n}$. Using the ratio test one has

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|3x+1|^4}{16}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|3x+1|^4}{16}.$$

If $\frac{|3x+1|^4}{16} < 1$, so if $-1 < x < \frac{1}{3}$, the series converges absolutely. If $\frac{|2x+1|^3}{8} > 1$, so if $x < -1$ or $x > \frac{1}{3}$, the series diverges. In the next step, we study the convergence of series at the end points: $x = -1$ and $x = \frac{1}{3}$.

$x = -1$ gives $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, so divergent ($p$-series with $p = 1$).

$x = \frac{1}{3}$ gives $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, so divergent ($p$-series with $p = 1$).

The interval of convergence therefore is $(-1, \frac{1}{3})$.

b) Let
$$f(x) = \sum_{n=1}^{\infty} \frac{(3x+1)^{4n}}{n\ 16^n}.$$

From part a) we know that f is differentiable on $(-1, \frac{1}{3})$ by differentiating of the series we get term by term:

$$f'(x) = \sum_{n=1}^{\infty} \frac{12n(3x+1)^{4n-1}}{n\ 16^n} = \sum_{n=1}^{\infty} \frac{12(3x+1)^{4n-1}}{16^n}.$$

Now put $x = 0$ to obtain:

$$f'(0) = \sum_{n=1}^{\infty} \frac{12}{16^n} = \sum_{n=1}^{\infty} 12\left(\frac{1}{16}\right)^n = 12 \times \frac{1/16}{1-(1/16)} = \frac{12}{15}.$$