Vrije Universiteit Amsterdam **May 29, 2020**
Analysis II - Final Exam **12:15 - 14:30**

Please motivate your answers! A correct answer without proper motivation will yield no points. Your hand-written answers need to submitted through Canvas, before 14:30 CEST in a single PDF file.

You have 135 minutes to complete the test. The result of this test will be your final grade.

Question 1 For each of the series below, determine whether the series convergences or diverges. In the case of convergence, determine whether the series converges absolutely or conditionally. `Justify your answer!`

(a) [5 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n\sqrt{n+3}}$

(b) [7 points] $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{\pi^n}$

Question 2 On the interval $[0,1]$, the sequence of functions $f_n(x)$ is defined by

$$f_n(x) = \frac{\arctan(nx)}{n}, n \geq 1.$$

(a) [7 points] Show that $\{f_n(x)\}$ converges uniformly on $[0,1]$.

(b) [3 points] Show that $\{f_n'(x)\}$ converges pointwise to a function $g(x)$ on $[0,1]$, but uniformly.

Question 3 [10 points] Calculate

$$\int_1^2 \sum_1^{\infty} \frac{1}{(n+x)^2}.$$

`Hint: First show that the series is uniformly convergent on` $[1,2]$.

Question 4 Consider the power series

$$\sum_{n=1}^{\infty} \frac{(1-2x)^{2n}}{n9^n}.$$

(a) [8 points] Determine the interval of convergence.

(b) [5 points] Let $g(x)$ be the sum of this power series in the open interval of convergence. Find $g'(0)$.

`Continue on the next page!`

Question 5 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^3 x_2 - 3x_2^5}{(x_1^2 + x_2^2)^{\frac{3}{2}}} & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

(a) [8 points] Is f continuous on $\mathbb{R}^2$?

(b) [5 points] Is f (totally) differentiable at $(0,0)$?

Question 6 [7 points] The function $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous second partial derivatives on $\mathbb{R}^2$. Let $z = f(x_1, x_2)$ with $x_1 = e^{u_1 + u_2}$, $x_2 = e^{u_1 u_2}$. Calculate $\frac{\partial z}{\partial u_1}$ and $\frac{\partial^2 z}{\partial u_2 \partial u_1}$ in terms of u_1 and u_2 and partial derivatives of f.

Question 7 [7 points] Find the minimum of the function $f(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2$ subject to the constraint $5x_1^2 - 4x_1x_2 + 5x_2^2 = 42$ using Lagrange multipliers.

Question 8 (a) [8 points] Let $G = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 - y^2 \leq 5, x \geq 6y, y \geq 0\}$. Calculate

$$\iint_G 4(x^2 - y^2)\frac{x - y}{x} dA.$$

(b) [8 points] Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, -x \leq y \leq x\}$. Calculate if possible,

$$\iint_D \frac{x}{(x^2 + y^2)^{\frac{1}{4}}} dA.$$