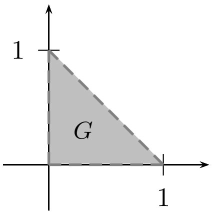
1.



a) $(0,0) \notin G$ and, therefore, is not interior. However, $(0,0) \in G'$ since, for each $r > 0$ and $x_1 = x_2 = \min\left\{\frac{r}{2}, \frac{1}{4}\right\}$, $(x_1, x_2) \in G$.

b) G is open. We show that each $(x_1, x_2) \in G$ is an interior point. Let $(x_1, x_2) \in G$ and let $r = \min\left\{x_1, x_2, \frac{1-x_1-x_2}{2}\right\}$. We show that $B((x_1, x_2); r) \subseteq G$. Let $(y_1, y_2) \in B((x_1, x_2); r)$. Then,

$$y_1 > x_1 - r \geq x_1 - x_1 = 0$$
$$y_2 > x_2 - r \geq x_2 - x_2 = 0$$
$$y_1 + y_2 < x_1 + r + x_2 + r = x_1 + x_2 + 2r \leq x_1 + x_2 + 1 - x_1 - x_2 = 1$$

and, thus, $(y_1, y_2) \in G$. This shows that $B((x_1, x_2); r) \subseteq G$ and, therefore, $(x_1, x_2) \in G^0$.

c) A set is compact if it is closed and bounded. The smallest closed set containing G is $\bar{G} = G \cup \partial G = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\}$. Moreover, $\bar{G} \subseteq \bar{B}((0,0), 1)$. Therefore, $\bar{G}$ is the smallest compact set containing G.

2. The function is not continuous at $(0,0)$ since $\lim_{(x_1,x_2)\to(0,0)} f(x_1, x_2) \neq 0$. To see this, let $x_2 = \lambda x_1$ with $\lambda \in \mathbb{R}$. Then,

$$\lim_{x_1\to 0} f(x_1, \lambda x_1) = \lim_{x_1\to 0} \frac{(1+\lambda^5)x_1^5 + \lambda^3 x_1^4}{(1+\lambda^4)x_1^4} = \lim_{x_1\to 0} \frac{(1+\lambda^5)x_1 + \lambda^3}{1+\lambda^4} = \frac{\lambda^3}{1+\lambda^4}$$

since we get a different limit for each value of λ, $\lim_{(x_1,x_2)\to(0,0)} f(x_1, x_2)$ does not exists. Therefore, f is not continuous on $\mathbb{R}^2$.

3. a) We use the definition.

$$f_{x_1}(0,0) = \lim_{h_1\to 0} \frac{f(h_1, 0) - f(0,0)}{h_1} = \lim_{h_1\to 0} \frac{\frac{h_1^2 0^3}{h_1^2+0^2} - 0}{h_1} = \lim_{h_1\to 0} \frac{0}{h_1^2} = 0$$

and

$$f_{x_2}(0,0) = \lim_{h_2\to 0} \frac{f(0, h_2) - f(0,0)}{h_2} = \lim_{h_2\to 0} \frac{\frac{0^2 h_2^3}{0^2+h_2^2} - 0}{h_2} = \lim_{h_2\to 0} \frac{0}{h_2} = 0.$$

b) Yes, it is. We show that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f(0,0) - f_{x_1}(0,0)h_1 - f_{x_2}(0,0)h_2}{\sqrt{h_1^2 + h_2^2}} = \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2 h_2^3}{(h_1^2 + h_2^2)^{\frac{3}{2}}} = 0.$$

Let $\epsilon > 0$ and $\delta = \sqrt{\epsilon}$. Let $(h_1, h_2) \in \mathbb{R}^2$ with $0 < ||(h_1, h_2)|| < \delta$. Then,

$$\left\|\frac{h_1^2 h_2^3}{(h_1^2+h_2^2)^{\frac{3}{2}}} - 0\right\| = \frac{h_1^2 |h_2|^3}{(h_1^2+h_2^2)^{\frac{3}{2}}} = \frac{h_1^2\left((h_2^2)^{\frac{1}{2}}\right)^3}{(h_1^2+h_2^2)^{\frac{3}{2}}} \leq \frac{h_1^2\left(h_1^2+h_2^2\right)^{\frac{3}{2}}}{(h_1^2+h_2^2)^{\frac{3}{2}}} = h_1^2 < \delta^2 = \epsilon.$$

This shows that the limit above converges to zero and, therefore, f is differentiable at $(0,0)$.

4. First, we calculate $\frac{\partial z}{\partial u_1}$ applying the chain rule. Since $x_1 = u_1^2 u_2$ and $x_2 = u_1 + 2u_2$,

$$\frac{\partial z}{\partial u_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_1} = 2u_1u_2\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}.$$

We now calculate $\frac{\partial^2 z}{\partial u_2 \partial u_1}$ applying the chain rule and the product rule when needed.

$$\begin{aligned}\frac{\partial^2 f}{\partial u_2 \partial u_1} &= 2u_1\frac{\partial f}{\partial x_1} + 2u_1u_2\left(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial x_1}{\partial u_2} + \frac{\partial^2 f}{\partial x_2 \partial x_1}\frac{\partial x_2}{\partial u_2}\right) + \frac{\partial^2 f}{\partial x_1 \partial x_2}\frac{\partial x_1}{\partial u_2} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial x_2}{\partial u_2} \\ &= 2u_1\frac{\partial f}{\partial x_1} + 2u_1u_2\left(u_1^2\frac{\partial^2 f}{\partial x_1^2} + 2\frac{\partial^2 f}{\partial x_2 \partial x_1}\right) + u_1^2\frac{\partial^2 f}{\partial x_1 \partial x_2} + 2\frac{\partial^2 f}{\partial x_2^2}.\end{aligned}$$

Since f has continuous second order partial derivatives, we have $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$ and

$$\frac{\partial^2 f}{\partial u_2 \partial u_1} = 2u_1\frac{\partial f}{\partial x_1} + 2u_1^3u_2\frac{\partial^2 f}{\partial x_1^2} + (4u_1u_2 + u_1^2)\frac{\partial^2 f}{\partial x_2 \partial x_1} + 2\frac{\partial^2 f}{\partial x_2^2}.$$

5. Let $h_1 = x_1 - 2$ and $h_2 = x_2 - 1$. Then,

$$p_2(x_1, x_2) = f(2,1) + \sum_{k=1}^{2}\frac{1}{k!}\left(h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2}\right)^k f(2,1).$$

Moreover,

$$\begin{aligned}
f(x_1,x_2) &= (2 + x_1 - 2x_2)^{-1} && \Rightarrow f(2,1) = \frac{1}{2} \\
\frac{\partial f}{\partial x_1}(x_1,x_2) &= -(2 + x_1 - 2x_2)^{-2} && \Rightarrow \frac{\partial f}{\partial x_1}(2,1) = -\frac{1}{4} \\
\frac{\partial f}{\partial x_2}(x_1,x_2) &= 2(2 + x_1 - 2x_2)^{-2} && \Rightarrow \frac{\partial f}{\partial x_2}(2,1) = \frac{1}{2} \\
\frac{\partial^2 f}{\partial x_1^2}(x_1,x_2) &= 2(2 + x_1 - 2x_2)^{-3} && \Rightarrow \frac{\partial^2 f}{\partial x_1^2}(2,1) = \frac{1}{4} \\
\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2) &= -4(2 + x_1 - 2x_2)^{-3} && \Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(2,1) = -\frac{1}{2} \\
\frac{\partial^2 f}{\partial x_2^2}(x_1,x_2) &= 8(2 + x_1 - 2x_2)^{-3} && \Rightarrow \frac{\partial^2 f}{\partial x_2^2}(2,1) = 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}p_2(x_1,x_2) &= f(2,1) + h_1\frac{\partial f}{\partial x_1}(2,1) + h_2\frac{\partial f}{\partial x_2}(2,1) + \frac{1}{2}h_1^2\frac{\partial^2 f}{\partial x_1^2}(2,1) + h_1h_2\frac{\partial^2 f}{\partial x_1 \partial x_2}(2,1) + \frac{1}{2}h_2^2\frac{\partial^2 f}{\partial x_2^2}(2,1) \\ &= \frac{1}{2} - \frac{h_1}{4} + \frac{h_2}{2} + \frac{h_1^2}{8} - \frac{h_1h_2}{2} + \frac{h_2^2}{2} \\ &= \frac{1}{2} - \frac{x_1 - 2}{4} + \frac{x_2 - 1}{2} + \frac{(x_1-2)^2}{8} - \frac{(x_1-2)(x_2-1)}{2} + \frac{(x_2-1)^2}{2}.\end{aligned}$$

6. a) Let $F(x,y,u,v) = (xe^y + uz - \cos v - 2, u\cos y + x^2 v - yz^2 - 1)$. We want to know if the equation
$$F(x,y,z,u,v) = (0,0)$$
has a solution for u and v as functions of x, y, and z in a neighborhood of $(2,0,1,1,0)$. We know that

- F has continuous partial derivatives,
- $F(2,0,1,1,0) = (2+1-1-2, 1+0-0-1) = (0,0)$.
- $\det\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}_{|(2,0,1,1,0)} = \det\begin{pmatrix} z & \sin v \\ \cos y & x^2 \end{pmatrix}_{|(2,0,1,1,0)} = \det\begin{pmatrix} 1 & 0 \\ 1 & 4 \end{pmatrix} = 4 \neq 0.$

By the implicit function theorem, there is a neighborhood of $(2,0,1,1,0)$ where u and v can be given as functions of x, y, and z.

b) We have $G(x,y,z) = F(x,y,z,u(x,y,z),v(x,y,z)) = 0$ for all (x,y,z) in a neighborhood of $(2,0,1)$. Thus,
$$\begin{aligned} 0 &= \frac{\partial G_1}{\partial z}(x,y,z) = \frac{\partial u}{\partial z}z + u + \sin(v)\frac{\partial v}{\partial z} \\ 0 &= \frac{\partial G_2}{\partial z}(x,y,z) = \frac{\partial u}{\partial z}\cos y + x^2\frac{\partial v}{\partial z} - 2yz \end{aligned}$$
Therefore,
$$\left.\begin{array}{l} \frac{\partial u}{\partial z}(2,0,1) + 1 + 0 = 0 \\ \frac{\partial u}{\partial z}(2,0,1) + 4\frac{\partial v}{\partial z}(2,0,1) - 0 = 0 \end{array}\right| \Leftrightarrow \begin{array}{l} \frac{\partial u}{\partial z}(2,0,1) = -1, \\ \frac{\partial v}{\partial z}(2,0,1) = \frac{1}{4}. \end{array}$$

7. Critical points: $\nabla f(x_1,x_2) = (0,0)$ if
$$\left.\begin{array}{r} 6x_1^2 - 6x_2 = 0 \\ -6x_1 + 6x_2 = 0 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} x_1^2 - x_1 = 0 \\ x_2 = x_1 \end{array}\right| \Leftrightarrow x_1 = x_2 = 0 \text{ or } x_1 = x_2 = 1.$$
Thus, we have two critical points: $(0,0)$ and $(1,1)$. To determine the nature of these points, we need the Hessian matrix.
$$H(x,y) = \begin{pmatrix} 12x_1 & -6 \\ -6 & 6 \end{pmatrix}.$$

- $(0,0)$: $H(0,0) = \begin{pmatrix} 0 & -6 \\ -6 & 6 \end{pmatrix} = A$. We have $\det(A_1) = 0$ and $\det(A_2) = -36 < 0$. Thus, $H(0,0)$ is indefinite and f has a saddle point at $(0,0)$.
- $(1,1)$: $H(1,1) = \begin{pmatrix} 12 & -6 \\ -6 & 6 \end{pmatrix} = A$. We have $\det(A_1) = 12 > 0$ and $\det(A_2) = 36 > 0$. Thus, $H(1,1)$is positive definite and f has a minimum at $(1,1)$.

8. Since our restriction describes the circle centered at $(0,0)$ with radius 3, we know that the function attains a maximum and a minimum under our constraint by the Extreme value theorem.

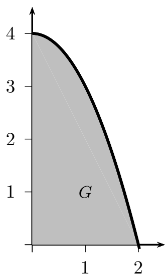
The Lagrange function is $\mathcal{L}(x_1,x_2,\lambda) = x_1^3 + 9x_1^2 + 6x_2^2 + \lambda(x_1^2 + x_2^2 - 9)$. The optima are obtained in the critical values of the Lagrange function. We get the system of equations

$$\left.\begin{array}{l} 3x_1^2 + 18x_1 + 2\lambda x_1 = 0 \\ 12x_2 + 2\lambda x_2 = 0 \\ x_1^2 + x_2^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 3x_1^2x_2 + 18x_1x_2 = -2\lambda x_1 x_2 \\ 12x_1x_2 = -2\lambda x_1x_2 \\ x_1^2 + x_2^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 3x_1^2 + 18x_1 + 2\lambda x_1 = 0 \\ 3x_1^2x_2 + 6x_1x_2 = 0 \\ x_1^2 + x_2^2 = 9 \end{array}\right|$$

$$\Leftrightarrow \left.\begin{array}{l} 3x_1^2 + 18x_1 + 2\lambda x_1 = 0 \\ x_1x_2(x_1+2) = 0 \\ x_1^2 + x_2^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} x_1 = 0, x_2 = \pm 3, \\ x_2 = 0, x_1 = \pm 3, \\ x_1 = -2, x_2 = \pm\sqrt{5} \end{array}\right| \text{ with } \begin{array}{l} f(0,-3) = f(0,3) = 54, \\ f(-3,0) = 54,\ f(3,0) = 108, \\ f(-2,\sqrt{5}) = f(-2,-\sqrt{5}) = 58. \end{array}$$

Then, the maximum is 108 and attained at $(3,0)$, and the minimum is 54 and attained at $(-3,0)$, $(0,-3)$, and $(0,3)$.

9. a)



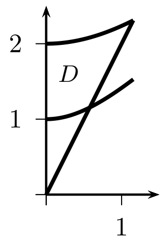
We integrate first with respect to x and, then, with respect to y. We have

$$G = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \sqrt{4-y}, 0 \le y \le 4\right\}.$$

Then,

$$\iint_G xe^{8y-y^2}\,dA = \int_0^4 \int_0^{\sqrt{4-y}} xe^{8y-y^2}\,dxdy = \frac{1}{2}\int_0^4 \left[x^2\right]_0^{\sqrt{4-y}} e^{8y-y^2}\,dy$$
$$= \frac{1}{2}\int_0^4 (4-y)e^{8y-y^2}\,dy = \frac{1}{4}\int_0^4 (8y-y^2)' e^{8y-y^2}\,dy$$
$$= \frac{1}{4}\left[e^{8y-y^2}\right]_0^4 = \frac{1}{4}\left(e^{16}-1\right).$$

b)



$D = \left\{(x,y) \in \mathbb{R}^2 \mid 1 \le y^2 - x^2 \le 4, 0 \le \frac{x}{y} \le \frac{1}{2}\right\}$. We use the substitution $u = y^2 - x^2$, $v = \frac{x}{y}$. Then, $D_{u,v} = \left\{(u,v) \in \mathbb{R}^2 | 1 \le u \le 4, 0 \le v \le \frac{1}{2}\right\}$ and

$$dudv = \left|\det\begin{pmatrix} -2x & 2y \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix}\right| dxdy = \left|2\frac{x^2}{y^2} - 2\right| dxdy = \left(2 - 2\frac{x^2}{y^2}\right) dxdy$$

with $2 - 2\frac{x^2}{y^2} \neq 0$ for each $(x_1,x_2) \in D^0$ since $0 < \frac{x}{y} < \frac{1}{2}$ implies $\frac{3}{2} < 2 - 2\frac{x^2}{y^2} < 2$. Then,

$$\iint_D \frac{x}{y}\,dA = \iint_D \frac{x}{y}\,\frac{2 - 2\frac{x^2}{y^2}}{2 - 2\frac{x^2}{y^2}}\,dA = \int_0^{\frac{1}{2}}\int_1^4 \frac{v}{2-2v^2}\,dudv = 3\int_0^{\frac{1}{2}} \frac{v}{2-2v^2}\,dv$$
$$= -\frac{3}{4}\int_0^{\frac{1}{2}} \frac{(2-2v^2)'}{2-2v^2}\,dv = -\frac{3}{4}\left[\ln|2-2v^2|\right]_0^{\frac{1}{2}} = \frac{3}{4}\left(\ln(2) - \ln\left(\frac{3}{2}\right)\right) = \frac{3}{4}\ln\left(\frac{4}{3}\right).$$