1. This is a nonlinear first-order differential equation, which can be solved using separation of the variables. This yields (we use $y = y(x)$):

$$\int y\,dy = \int \cos x\,dx \Longrightarrow \frac{1}{2}y^2 = \sin x + C,\ C \in \mathbb{R}.$$

So the general solution is either

$$y(x) = \sqrt{2\sin x + 2C}, C \in \mathbb{R}$$

or

$$y(x) = -\sqrt{2\sin x + 2C}, C \in \mathbb{R}.$$

Substitution of the initial value shows that we need the +-sign and that

$$2 = y(\pi) = \sqrt{2C}, \qquad \text{so} \qquad C = 2.$$

So the solution of this initial value problem is $y(x) = \sqrt{2\sin x + 4}$.

2. First solve the homogeneous equation. Substitution of $y(x) = e^{\lambda x}$ gives the auxiliary equation $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0$, with only one solution $\lambda = -\frac{1}{2}$. So the general solution of the homogeneous equation is

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}, c_1, c_2 \in \mathbb{R}.$$

A particular solution will be of the form $y(x) = Ax + B + Ce^x$. Then we have $y'(x) = A + Ce^x$ and $y''(x) = Ce^x$. Substitution in the nonhomogeneous equation yields

$$Ax + (4A + B) + 9Ce^x = x + e^x, \text{ so } A = 1, B = -4 \text{ and } C = \frac{1}{9}.$$

The general solution for this inhomogeneous differential equation is therefore:

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x} + x - 4 + \frac{1}{9}e^x, c_1, c_2 \in \mathbb{R}.$$

3. a) First consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n + \ln n} \right| = \sum_{n=1}^{\infty} \frac{1}{n + \ln n}.$$

Since $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$, we can compare the general term with $\frac{1}{n}$. We find

$$\lim_{n\to\infty} \frac{1}{n + \ln n} : \frac{1}{n} = \lim_{n\to\infty} \frac{1}{1 + \frac{\ln n}{n}} = 1,$$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series with $p = 1$), the series $\sum\limits_{n=1}^{\infty} \frac{1}{n+\ln n}$ is also divergent, according to the limit comparison test. So there is no absolute convergence. Now apply the alternating series test: (i) the series is alternating, (ii) the general term tends to 0 and (iii) the sequence $\left\{\frac{1}{n+\ln n}\right\}$ is decreasing. So the series is convergent, but not absolutely convergent, so it is conditionally convergent.

b) Apply the ratio test and use the formula $(n+1)! = (n+1)(n)!$

$$\lim_{n\to\infty} \frac{e^{(n+1)^2}}{(n+1)!} : \frac{e^{n^2}}{n!} = \lim_{n\to\infty} \frac{e^{2n+1}}{n+1} = \infty > 1$$

which means that the series is divergent.

4. a) The statement is false. Choose $a_n = \frac{(-1)^n}{n}$. Then $a_{2n} = \frac{(-1)^{2n}}{2n} = \frac{1}{2n}$. The series $\sum_{n=1}^{\infty} a_n$ converges (alternating series test), while $\sum_{n=1}^{\infty} a_{2n}$ diverges (p-series with $p = 1$).

b) The statement is false. Choose $b_n = 1 - (-1)^n$, which means that $\{b_n\} = \{2, 0, 2, 0, 2, 0, \ldots\}$. Clearly $\lim_{n\to\infty} b_n \neq 0$, so $\sum_{n=1}^{\infty} b_n$ diverges. However $b_{2n} = 0$ for all n, so $\sum_{n=1}^{\infty} b_{2n}$ converges.

5. a) First, we will prove that $f(x)$ is the point wise limit. Choose an arbitrary $x_0 \in [-a, a]$. Then, we have

$$\lim_{n\to\infty} \frac{x_0^2}{n} = 0 = f(x_0).$$

Now, we will prove uniform convergence. Let $\varepsilon > 0$ be given. Choose $n^* = \frac{a^2}{\varepsilon}$. Then, for all $n > n^*$ we find that for each $x \in [-a, a]$:

$$|f_n(x) - f(x)| = \left| \frac{x^2}{n} - 0 \right| \leq \frac{a^2}{n} < \frac{a^2}{n^*} = \varepsilon.$$

b) The point wise limit is still $f(x) = 0$, but the sequence is not uniformly convergent on $\mathbb{R}$. Choose $\varepsilon = \frac{1}{2}$ and $x_n = \sqrt{n}$. Then

$$|f_n(x_n) - f(x_n)| = \left| \frac{(\sqrt{n})^2}{n} - 0 \right| = 1 \geq \frac{1}{2},$$

so the sequence $\{f_n(x)\}$ does not converge uniformly on $\mathbb{R}$.

6. a) Define $f_n(x) = xe^{-nx}$ for all $n \geq 1$. Then, $f_n'(x) = (1 - nx)e^{-nx}$. So $f_n'(x) = 0$ for $x = \frac{1}{n}$ and $f_n(x)$ are decreasing for $x > \frac{1}{n}$. If n is large enough, $a > \frac{1}{n}$, so for n large enough, $f_n(x)$ has its maximum value at $x = a$. Now, we apply the Weierstrass M-test. On $[a, \infty)$, we have for large n that $f_n(x) = xe^{-nx} \leq ae^{-an} = M_n$ and since

$$\sum_{n=0}^{\infty} ae^{-an} = a \sum_{n=0}^{\infty} \left(e^{-a}\right)^n = \frac{a}{1 - e^{-a}} \text{ (is convergent)}$$

the Weierstrass M-test concludes that $\sum_{n=0}^{\infty} xe^{-nx}$ is uniform convergent on $[a, \infty)$.

b) Let $f(x) = \sum_{n=0}^{\infty} xe^{-nx}$. Then $f(0) = 0$, while for $x > 0$ we have

$$f(x) = \sum_{n=0}^{\infty} xe^{-nx} = \sum_{n=0}^{\infty} x\left(e^{-x}\right)^n = \frac{x}{1-e^{-x}}.$$

Since $f_n(x) = xe^{-nx}$ is continuous on $[0,\infty)$ for all n, we should find a continuous sum function $f(x)$ on $[0,\infty)$ in case of uniform convergence. However,

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x}{1-e^{-x}} = 1 \neq 0 = f(0) \text{ (use l'Hospital)},$$

so this series is not uniform convergent on $[0,\infty)$.

7. a) Suppose $a_n = \dfrac{(2x-1)^{3n}}{n\,8^n}$ and apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|2x-1|^3}{8}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|2x-1|^3}{8}.$$

If $\frac{|2x-1|^3}{8} < 1$, so if $-\frac{1}{2} < x < \frac{3}{2}$, the series converges absolutely. If $\frac{|2x-1|^3}{8} > 1$, so if $x < -\frac{1}{2}$ or $x > \frac{3}{2}$, the series diverges. Now consider the endpoints $x = -\frac{1}{2}$ and $x = \frac{3}{2}$ separately:

$x = \frac{3}{2}$ gives $\sum_{n=1}^{\infty} \frac{1}{n}$, so divergent ($p$-series with $p = 1$).

$x = -\frac{1}{2}$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, so a convergent series, according to the alternating series test.

The interval of convergence therefore is $[-\frac{1}{2}, \frac{3}{2})$.

b) Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(2x-1)^{3n}}{n\,8^n}.$$

In part a) we have proven that the interval of convergence is $[-\frac{1}{2}, \frac{3}{2})$. Then, f is differentiable on $(-\frac{1}{2}, \frac{3}{2})$ and its derivative can be found by differentiating term by term:

$$f'(x) = \sum_{n=1}^{\infty} \frac{6n(2x-1)^{3n-1}}{n\,8^n} = \sum_{n=1}^{\infty} \frac{6(2x-1)^{3n-1}}{8^n}.$$

Now substitute $x = 0$ to obtain:

$$f'(0) = \sum_{n=1}^{\infty} \frac{6(-1)^{3n-1}}{8^n} = \sum_{n=1}^{\infty} -6\left(-\frac{1}{8}\right)^n = -6 \times \frac{-1/8}{1-(-1/8)} = \frac{6}{9} = \frac{2}{3}.$$

8. To simplify the calculations, we substitute $t = x + 1$ and find a series representation of $\frac{5}{2-3(t-1)} = \frac{5}{5-3t}$ in powers of t (using the geometric series):

$$\frac{5}{5-3t} = \frac{1}{1-3t/5} = \sum_{n=0}^{\infty}\left(\frac{3}{5}t\right)^n,$$

which converges for $|\frac{3}{5}t| < 1$, so for $|t| < \frac{5}{3}$. Now replace t by $x+1$ to get the desired result

$$\frac{5}{2-3x} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n (x+1)^n,$$

which converges for $|x+1| < \frac{5}{3}$, so for $-\frac{8}{3} < x < \frac{2}{3}$.

9. We start with the well known Maclaurin series for e^t:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \text{ for all } t \in \mathbb{R}, \text{ which leads to}$$

$$t^2 e^{5t^2} = t^2 \sum_{n=0}^{\infty} \frac{(5t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n t^{2n+2}}{n!}.$$

This power series converges uniformly to $t^2 e^{5t^2}$ on any bounded subset of $\mathbb{R}$. So we can interchange series and integral to obtain

$$\begin{aligned} K(x) &= \int_0^x t^2 e^{5t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} \frac{5^n t^{2n+2}}{n!}\, dt = \sum_{n=0}^{\infty} \frac{5^n}{n!} \int_0^x t^{2n+2}\, dt = \\ &= \sum_{n=0}^{\infty} \frac{5^n}{n!} \frac{t^{2n+3}}{2n+3}\Big|_0^x = \sum_{n=0}^{\infty} \frac{5^n}{n!} \frac{x^{2n+3}}{2n+3}, \end{aligned}$$

which is convergent to $K(x)$ for all $x \in \mathbb{R}$.