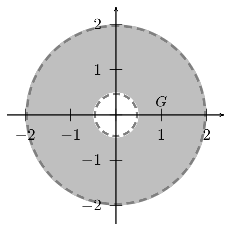
1. a)



b) $\text{int}(G) = G$,
$\partial G = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = \frac{1}{4}\} \cup \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 = 4\}$, and
$G' = \{(x_1, x_2) \in \mathbb{R}^2 | \frac{1}{4} \leq x_1^2 + x_2^2 \leq 4\}$.

c) Since $G = \text{int}G$, we have that G is open.

d) G is not convex since $(1,0), (-1,0) \in G$ and, for $\lambda = \frac{1}{2} \in [0,1]$,

$$\frac{1}{2}(1,0) + \frac{1}{2}(-1,0) = (0,0) \notin G.$$

G is connected because for any $(x_1, x_2), (y_1, y_2) \in G$ either the segment line joining (x_1, x_2) and (y_1, y_2) is contained in G, or we can we can find a polygonal line connecting them both that is contained in G by using (some of) the points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, and $(\frac{1}{2}, \frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$.

2. a) Let $\varepsilon > 0$ and $\delta = \min\{1, \frac{\varepsilon}{2}\}$. For $(x_1, x_2) \in \mathbb{R}^2$ with $0 < ||(x_1, x_2)|| < \delta$, it follows

$$||f(x_1, x_2) - f(0,0)|| = \left| \frac{x_1^3(x_2+1)}{x_1^2 + x_2^2} \right| = \frac{x_1^2 |x_1||x_2+1|}{x_1^2 + x_2^2} \leq \frac{(x_1^2 + x_2^2)|x_1||x_2+1|}{x_1^2 + x_2^2}$$
$$= |x_1||x_2+1| \leq |x_1|(|x_2|+1) < 2\delta \leq \varepsilon.$$

This shows that f is continuous at $(0,0)$.

b) We use the definition.

$$f_{x_1}(0,0) = \lim_{h_1 \to 0} \frac{f(h_1, 0) - f(0,0)}{h_1} = \lim_{h_1 \to 0} \frac{\frac{h_1^3(0+1)}{h_1^2 + 0^2} - 0}{h_1} = \lim_{h_1 \to 0} \frac{h_1^3}{h_1^3} = 1$$

and

$$f_{x_2}(0,0) = \lim_{h_2 \to 0} \frac{f(0, h_2) - f(0,0)}{h_2} = \lim_{h_2 \to 0} \frac{\frac{0^3(h_2+1)}{0^2 + h_2^2} - 0}{h_2} = \lim_{h_2 \to 0} \frac{0}{h_2} = 0.$$

c) To show that f is not differentiable at $(0,0)$, we see that the limit below does not converge to zero.

$$\lim_{(h_1,h_2) \to (0,0)} \frac{f(h_1, h_2) - f(0,0) - f_{x_1}(0,0)h_1 - f_{x_2}(0,0)h_2}{\sqrt{h_1^2 + h_2^2}}$$
$$= \lim_{(h_1,h_2) \to (0,0)} \frac{\frac{h_1^3(h_2+1)}{h_1^2 + h_2^2} - 0 - h_1 - 0}{\sqrt{h_1^2 + h_2^2}} = \lim_{(h_1,h_2) \to (0,0)} \frac{\frac{h_1^3(h_2+1) - h_1^3 - h_1 h_2^2}{h_1^2 + h_2^2}}{\sqrt{h_1^2 + h_2^2}}$$
$$= \lim_{(h_1,h_2) \to (0,0)} \frac{h_1^3 h_2 - h_1 h_2^2}{(h_1^2 + h_2^2)^{\frac{3}{2}}}$$

Take $h_2 = h_1$. Then,

$$\lim_{h_1\to 0}\frac{h_1^4-h_1^3}{(2h_1^2)^{\frac{3}{2}}} = \lim_{h_1\to 0}\frac{h_1^3(h_1-1)}{2^{\frac{3}{2}}|h_1|^3} = \begin{cases} -2^{-\frac{3}{2}} & \text{when } h_1\to 0^+, \\ 2^{-\frac{3}{2}} & \text{when } h_1 \to 0^-. \end{cases}$$

Therefore, the limit does not exist and f is not differentiable at $(0,0)$.

d) Since f is not differentiable at $(0,0)$, we need to use the definition.

$$\frac{\partial f}{\partial \mathbf{u}}(0,0) = \lim_{h\to 0}\frac{f(hu_1,hu_2)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3u_1^3(hu_2+1)}{h^2(u_1^2+u_2^2)}}{h}$$
$$\underset{||\mathbf{u}||=1}{=} \lim_{h\to 0}\frac{h^3u_1^3(hu_2+1)}{h^3} = \lim_{h\to 0} u_1^3(hu_2+1) = u_1^3.$$

3. a) g has continuous first order partial derivatives and $\det(Dg(u_1,u_2)) = \det\begin{pmatrix} 2u_1 & -2u_2 \\ u_2 & u_1\end{pmatrix} = 2(u_1^2+u_2^2)\neq 0$ for all $(u_1,u_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. By the Inverse function theorem, g has a (local) inverse for any point in $\mathbb{R}^2\setminus\{(0,0)\}$.

b) First, we calculate $\frac{\partial z}{\partial u_1}$ applying the chain rule. Let $x_1 = u_1^2 - u_2^2$ and $x_2 = u_1u_2$

$$\frac{\partial z}{\partial u_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial u_1} = 2u_1\frac{\partial f}{\partial x_1} + u_2\frac{\partial f}{\partial x_2}.$$

We now calculate $\frac{\partial^2 z}{\partial u_1^2}$ applying the chain rule and the product rule when needed.

$$\frac{\partial^2 f}{\partial u_1^2} = 2\frac{\partial f}{\partial x_1} + 2u_1\left(\frac{\partial^2 f}{\partial x_1^2}\frac{\partial x_1}{\partial u_1} + \frac{\partial^2 f}{\partial x_2\partial x_1}\frac{\partial x_2}{\partial u_1}\right) + u_2\left(\frac{\partial^2 f}{\partial x_1\partial x_2}\frac{\partial x_1}{\partial u_1} + \frac{\partial^2 f}{\partial x_2^2}\frac{\partial x_2}{\partial u_1}\right)$$
$$= 2\frac{\partial f}{\partial x_1} + 2u_1\left(2u_1\frac{\partial^2 f}{\partial x_1^2} + u_2\frac{\partial^2 f}{\partial x_2\partial x_1}\right) + u_2\left(2u_1\frac{\partial^2 f}{\partial x_1\partial x_2} + u_2\frac{\partial^2 f}{\partial x_2^2}\right)$$

Since f has continuous second order partial derivatives, we have $\frac{\partial^2 f}{\partial x_1\partial x_2} = \frac{\partial^2 f}{\partial x_2\partial x_1}$ and

$$\frac{\partial^2 f}{\partial u_1^2} = 2\frac{\partial f}{\partial x_1} + 4u_1^2\frac{\partial^2 f}{\partial x_1^2} + u_2^2\frac{\partial^2 f}{\partial x_2^2} + 4u_1u_2\frac{\partial^2 f}{\partial x_2\partial x_1}$$

4. a) $\nabla f(x_1,x_2) = (2x_2 - 4x_1^3 - 2x_1, 2x_1 - 2x_2)$ and $Hf(x_1,x_2) = \begin{pmatrix} -12x_1^2-2 & 2 \\ 2 & -2\end{pmatrix} = A.$

b) $|A_1| = -12x_1^2 - 2 < 0$ for all $(x_1,x_2)\in\mathbb{R}^2$ and $|A_2| = 24x_1^2 \geq 0$ for all $(x_1,x_2)\in\mathbb{R}^2$. Then, $Hf(x_1,x_2)$ is negative semi-definite in $\mathbb{R}^2$ and f is concave.

c) Critical values: $\nabla f(x_1,x_2) = (0,0)$ if

$$\left.\begin{array}{l} 2x_2 - 4x_1^3 - 2x_1 = 0 \\ 2x_1 - 2x_2 = 0 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} -2x_1^3 = 0 \\ x_1 = x_2 \end{array}\right| \Leftrightarrow x_1 = x_2 = 0.$$

Since f is concave, f has a global maximum at $(0,0)$.

5. Since our restriction describes an ellipse, we know that the function attains a maximum and a minimum under our constraint by the Extreme value theorem. The Lagrange function is

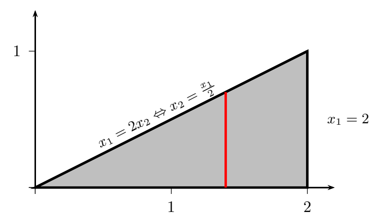
$\mathcal{L}(x_1, x_2, \lambda) = x_1 - x_2 + \lambda(x_1^2 + 2x_2^2 - 74^2)$. The optima are obtained in the critical values of the Lagrange function. We get the system of equations

$$\left.\begin{array}{l} 1 + 2\lambda x_1 = 0 \\ -1 + 4\lambda x_2 = 0 \\ x_1^2 + 2x_2^2 = 74^2 \end{array}\right| \underset{\lambda \neq 0}{\Leftrightarrow} \left.\begin{array}{l} -x_1 = \frac{1}{2\lambda} \\ 2x_2 = \frac{1}{2\lambda} \\ x_1^2 + 2x_2^2 = 74^2 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} -x_1 = \frac{1}{2\lambda} \\ x_1 = -2x_2 \\ x_1^2 + 2x_2^2 = 74^2 \end{array}\right| \Leftrightarrow$$

$$\left.\begin{array}{l} -x_1 = \frac{1}{2\lambda} \\ x_1 = -2x_2 \\ 4x_2^2 + 2x_2^2 = 74^2 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} -x_1 = \frac{1}{2\lambda} \\ x_1 = -2x_2 \\ x_2^2 = \frac{2}{3}37^2 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} -x_1 = \frac{1}{2\lambda} \\ x_1 = \mp 2\sqrt{\frac{2}{3}}37 \\ x_2 = \pm\sqrt{\frac{2}{3}}37 \end{array}\right|$$

Then, the maximum is $37\sqrt{6}$ and attained at $\left(2\sqrt{\frac{2}{3}}37, -\sqrt{\frac{2}{3}}37\right)$, and the minimum is $-37\sqrt{6}$ and attained at $\left(-2\sqrt{\frac{2}{3}}37, \sqrt{\frac{2}{3}}37\right)$.

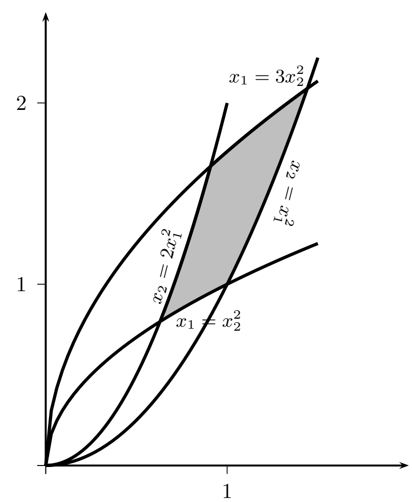
6. We change the order of integration.



$$\int_0^1 \left[\int_{2x_2}^2 \frac{96x_2^2}{x_1^4+1} dx_1\right] dx_2 = \int_0^2 \left[\int_0^{\frac{x_1}{2}} \frac{96x_2^2}{x_1^4+1} dx_2\right] dx_1 = \int_0^2 \left[\frac{32x_2^3}{x_1^4+1}\right]_0^{\frac{x_1}{2}} dx_1$$

$$= \int_0^2 \frac{4x_1^3}{x_1^4+1} dx_1 = \int_0^2 \frac{(x_1^4+1)'}{x_1^4+1} dx_1 = \left[\ln(x_1^4+1)\right]_0^2 = \ln(17).$$

7.



We use the substitution $u_1 = \frac{x_1}{x_2^2}$, $u_2 = \frac{x_2}{x_1^2}$. Then,

$$\left|\det\begin{pmatrix} \frac{1}{x_2^2} & -\frac{2x_1}{x_2^3} \\ -\frac{2x_2}{x_1^3} & \frac{1}{x_1^2} \end{pmatrix}\right| = \frac{3}{x_1^2 x_2^2} \neq 0$$

for every $(x_1, x_2) \in D$ and $du_1 du_2 = \frac{3dx_1dx_2}{x_1^2x_2^2}$. Moreover,

$$D_{u_1,u_2} = \{(u_1, u_2) \in \mathbb{R}^2 | 1 \leq u_1 \leq 3, 1 \leq u_2 \leq 2\}$$

and $x_1x_2 = \frac{1}{u_1u_2}$.

Then,

$$\iint_D x_1x_2 dA = \frac{1}{3}\iint_D x_1^3x_2^3 \frac{3dA}{x_1^2x_2^2} = \frac{1}{3}\int_1^2 \left[\int_1^3 \frac{du_1}{u_1^3u_2^3}\right] du_2 = \frac{1}{3}\int_1^2 \left[\frac{-1}{2u_1^2u_2^3}\right]_1^3 du_2$$

$$= \frac{1}{3}\int_1^2 \frac{4}{9u_2^3} du_2 = \frac{4}{27}\left[\frac{-1}{2u_2^2}\right]_1^2 = \frac{1}{18}.$$

8. Let

$$D_m = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^2 \mid \frac{1}{m^2} \leq x_1^2 + x_2^2 \leq 1, x_1 \geq 0, x_2 \geq 0, 0 \leq x_3 \leq 1 \right\}.$$

Cylindrical coordinates are given by $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, and $x_3 = x_3$, where $0 \leq r \leq 1$, $0 \leq \phi \leq \frac{\pi}{2}$, and $0 \leq x_3 \leq 1$; further, $dV = r dr d\phi dx_3$. Then,

$$\begin{aligned}
\int\int\int_D \frac{x_3}{\sqrt[6]{x_1^2 + x_2^2}} dV &= \lim_{m\to\infty} \int\int\int_{D_m} \frac{x_3}{\sqrt[6]{x_1^2 + x_2^2}} dV = \lim_{m\to\infty} \int_{\frac{1}{m}}^{1} \left[\int_0^{\frac{\pi}{2}} \left[\int_0^1 \frac{x_3}{r^{\frac{1}{3}}} r dx_3 \right] d\phi \right] dr \\
&= \lim_{m\to\infty} \int_{\frac{1}{m}}^{1} \left[\int_0^{\frac{\pi}{2}} \left[\frac{x_3^2}{2} r^{\frac{2}{3}} \right]_0^1 d\phi \right] dr = \lim_{m\to\infty} \frac{1}{2} \int_{\frac{1}{m}}^{1} \left[\int_0^{\frac{\pi}{2}} r^{\frac{2}{3}} d\phi \right] dr \\
&= \lim_{m\to\infty} \frac{\pi}{4} \int_{\frac{1}{m}}^{1} r^{\frac{2}{3}} dr = \lim_{m\to\infty} \frac{\pi}{4} \left[\frac{3}{5} r^{\frac{5}{3}} \right]_{\frac{1}{m}}^{1} = \lim_{m\to\infty} \frac{3\pi}{20} \left(1 - \left(\frac{1}{m} \right)^{\frac{5}{3}} \right) = \frac{3\pi}{20}.
\end{aligned}$$