1. a) $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$

   $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$

   b) The function is not totally differentiable in $(0,0)$. To see this, we have to show that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} \neq 0.$$

   Take, for instance, $k = h$ an let $h \to 0$. Then,

$$\lim_{h\to 0} \frac{f(h,h)-f(0,0)-hf_x(0,0)-hf_y(0,0)}{\sqrt{2h^2}} = \lim_{h\to 0} \frac{\frac{h^2}{\sqrt{2h^2}}}{\sqrt{2h^2}} = \lim_{h\to 0} \frac{h^2}{2h^2} = \frac{1}{2} \neq 0.$$

   c) We use the definition of directional derivative:

$$D_{\vec{u}}f(0,0) = \lim_{h\to 0} \frac{f(hu_1,hu_2)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2u_1u_2}{\sqrt{h^2(u_1^2+u_2^2)}}}{h} = \lim_{h\to 0} \frac{hu_1u_2}{|h|}.$$

   This limit only converges if either $u_1 = 0$ or $u_2 = 0$, i.e., for $\vec{u} \in \{(1,0),(-1,0),(0,1),(0,-1)\}$.

2. We have $z = f(u,v) = f(xy, \frac{y}{x})$. Using the chain rule, we have

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = xf_u + \frac{1}{x}f_v.$$

   Using the chain rule and the product rule when needed, and $f_{uv} = f_{vu}$, we have

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= f_u + x\left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial x}\right) - \frac{1}{x^2}f_v + \frac{1}{x}\left(\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial x}\right)\\ &= f_u + x\left(yf_{uu} - \frac{y}{x^2}f_{uv}\right) - \frac{1}{x^2}f_v + \frac{1}{x}\left(yf_{vu} - \frac{y}{x^2}f_{vv}\right)\\ &= f_u - \frac{1}{x^2}f_v + xyf_{uu} - \frac{y}{x^3}f_{vv}.\end{aligned}$$

3. We use Taylor's series. Let $h = x-1$ and $k = y+2$. Then,

$$f(x,y) = f(1,-2) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^k f(1,-2).$$

Note that $\dfrac{\partial^k f}{\partial x^l \partial y^{k-l}}(x,y) = 0$ for every $k \geq 4$ and for every $l \in \{0,\ldots,k\}$. Moreover,

$$\begin{aligned}
& & & f(1,-2) = 0 \\
&\frac{\partial f}{\partial x}(x,y) = 3x^2 - 6x - 63 + 2y^2 - 29y & \Rightarrow & \frac{\partial f}{\partial x}(1,-2) = 0 \\
&\frac{\partial f}{\partial y}(x,y) = 4xy - 29x - 4y + 29 & \Rightarrow & \frac{\partial f}{\partial y}(1,-2) = 0 \\
&\frac{\partial^2 f}{\partial x^2}(x,y) = 6x - 6 & \Rightarrow & \frac{\partial^2 f}{\partial x^2}(1,-2) = 0 \\
&\frac{\partial^2 f}{\partial x \partial y}(x,y) = 4y - 29 & \Rightarrow & \frac{\partial^2 f}{\partial x \partial y}(1,-2) = -37 \\
&\frac{\partial^2 f}{\partial y^2}(x,y) = 4x - 4 & \Rightarrow & \frac{\partial^2 f}{\partial y^2}(1,-2) = 0 \\
&\frac{\partial^3 f}{\partial x^3}(x,y) = 6 & \Rightarrow & \frac{\partial^3 f}{\partial x^3}(1,-2) = 6 \\
&\frac{\partial^3 f}{\partial y \partial x^2}(x,y) = 0 & \Rightarrow & \frac{\partial^3 f}{\partial y \partial x^2}(1,-2) = 0 \\
&\frac{\partial^3 f}{\partial x \partial y^2}(x,y) = 4 & \Rightarrow & \frac{\partial^3 f}{\partial x \partial y^2}(1,-2) = 4 \\
&\frac{\partial^3 f}{\partial y^3}(x,y) = 0 & \Rightarrow & \frac{\partial^3 f}{\partial y^3}(1,-2) = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x,y) &= f(1,-2) + \sum_{k=1}^{\infty} \frac{1}{k!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^k f(1,-2) \\
&= f(-1,2) + h\frac{\partial f}{\partial x}(-1,2) + k\frac{\partial f}{\partial y}(-1,2) \\
&\quad + \frac{1}{2}h^2\frac{\partial^2 f}{\partial x^2}(-1,2) + hk\frac{\partial^2 f}{\partial x \partial y}(-1,2) + \frac{1}{2}k^2\frac{\partial^2 f}{\partial y^2}(-1,2) \\
&\quad + \frac{1}{6}h^3\frac{\partial^3 f}{\partial x^3}(-1,2) + \frac{1}{2}h^2k\frac{\partial^3 f}{\partial x^2 \partial y}(-1,2) + \frac{1}{2}hk^2\frac{\partial^3 f}{\partial x \partial y^2}(-1,2) + \frac{1}{6}k^3\frac{\partial^3 f}{\partial y^3}(-1,2) \\
&= -37hk + h^3 + 2hk^2 \\
&= (x-1)^3 + 2(x-1)(y+2)^2 - 37(x-1)(y+2).
\end{aligned}$$

4. First, we compute the critical points, that is, the points where $\nabla f(x,y) = (0,0)$.

$$\nabla f(x,y) = \langle -6x^2 + 6y, 6y + 6x \rangle = \langle 0,0 \rangle \Leftrightarrow \left. \begin{array}{l} x^2 - y = 0 \\ y = -x \end{array} \right| \Leftrightarrow \left. \begin{array}{l} x(x+1) = 0 \\ y = -x \end{array} \right|$$

Therefore, $(x,y) = (0,0)$ or $(x,y) = (-1,1)$. To study the nature of these points, we look at the Hessian matrix: $Hf(x,y) = \begin{pmatrix} -12x & 6 \\ 6 & 6 \end{pmatrix}$.

For $(0,0)$, we have $Hf(0,0) = \begin{pmatrix} 0 & 6 \\ 6 & 6 \end{pmatrix} = A$, which is indefinite since $|A_1| = 0$ and $|A_2| = |A| = -36$. Therefore, f has a saddle point at $(0,0)$.

For $(-1,1)$, we have $Hf(-1,1) = \begin{pmatrix} 12 & 6 \\ 6 & 6 \end{pmatrix} = A$, which is positive definite since $|A_1| = 12$ and $|A_2| = |A| = 36$. Therefore, f has a minimum at $(-1,1)$.

5. We have to solve

$$\begin{array}{ll} \min & x^3 + 9x^2 + 6y^2 \\ \text{s.t.} & x^2 + y^2 = 9. \end{array}$$

(i) Lagrange function: $\mathcal{L}(x,y,\lambda) = x^3 + 9x^2 + 6y^2 + \lambda(x^2 + y^2 - 9)$.

(ii) $\nabla \mathcal{L}(x,y,\lambda) = \vec{0}$.

$$\left.\begin{array}{l} 3x^2 + 18x + 2\lambda x = 0 \\ 12y + 2\lambda y = 0 \\ x^2 + y^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 3x^2y + 18xy + 2\lambda xy = 0 \\ 12xy + 2\lambda xy = 0 \\ x^2 + y^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 6y + \lambda y = 0 \\ 3x^2y + 6xy = 0 \\ x^2 + y^2 = 9 \end{array}\right|$$

$$\Leftrightarrow \left.\begin{array}{l} 6y + \lambda y = 0 \\ xy(x+2) = 0 \\ x^2 + y^2 = 9 \end{array}\right|.$$

Then, we have the solutions: $(0,3)$, $(0,-3)$, $(3,0)$, $(-3,0)$, $(-2,\sqrt{5})$, $(-2,-\sqrt{5})$.
Alternative:

$$\left.\begin{array}{l} 3x^2 + 18x + 2\lambda x = 0 \\ 12y + 2\lambda y = 0 \\ x^2 + y^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 3x^2 + 18x + 2\lambda x = 0 \\ y(6+\lambda) = 0 \\ x^2 + y^2 = 9 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 3x^2 + 18x + 2\lambda x = 0 \\ y = 0 \text{ or } \lambda = -6 \\ x^2 + y^2 = 9 \end{array}\right|$$

For $y = 0$, we have the solutions: $(3,0)$, $(-3,0)$. For $\lambda = -6$, we have $3x^2 + 18x - 12x = 3x(x+2) = 0$; then, the solutions are: $(0,3)$, $(0,-3)$, $(-2,\sqrt{5})$, $(-2,-\sqrt{5})$.

(iii) Solution.
We evaluate $f(x,y) = x^3 + 9x^2 + 6y^2$ in all critical points and we have

$$f(0,3) = f(0,-3) = 54, \ f(3,0) = 108, \ f(-3,0) = 54, \ f(-2,\sqrt{5}) = f(-2,-\sqrt{5}) = 58.$$

Therefore, the minimum value is 54 and is attained at $(0,3)$, $(0,-3)$, and $(-3,0)$.

6. a) Let $F(x,y,u,v) = (2xu^3v - yv - 1, y^3v + x^5u^2 - 2)$. We want to know if the equation

$$F(x,y,u,v) = (0,0)$$

has a solution for u and v as functions of x and y in a neighborhood of $(1,1,1,1)$. We know that

- F has continuous partial derivatives,

- $F(1,1,1,1) = (2-1-1, 1+1-2) = (0,0)$.
- $\det\begin{pmatrix} \frac{\partial F_1}{\partial u}(1,1,1,1) & \frac{\partial F_1}{\partial v}(1,1,1,1) \\ \frac{\partial F_2}{\partial u}(1,1,1,1) & \frac{\partial F_2}{\partial v}(1,1,1,1) \end{pmatrix} = \det\begin{pmatrix} 6xu^2v & 2xu^3 - y \\ 2x^5u & y^3 \end{pmatrix}_{|(1,1,1,1)} = \det\begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix} = 4 \neq 0$.

  By the implicit function theorem, there is a neighborhood of $(1,1,1,1)$ where u and v can be given as functions of x and y.

b) We have $F(x,y,u(x,y),v(x,y)) = 0$ for all (x,y) in a neighborhood of $(1,1)$. Thus,

$$\begin{aligned}
0 &= \frac{\partial F_1}{\partial x}(x,y,u(x,y),v(x,y)) = 2u^3v + 6xu^2v\frac{\partial u}{\partial x} + 2xu^3\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial x} \\
0 &= \frac{\partial F_1}{\partial y}(x,y,u(x,y),v(x,y)) = 6xu^2v\frac{\partial u}{\partial y} + 2xu^3\frac{\partial v}{\partial y} - v - y\frac{\partial v}{\partial y} \\
0 &= \frac{\partial F_2}{\partial x}(x,y,u(x,y),v(x,y)) = y^3\frac{\partial v}{\partial x} + 5x^4u^2 + 2x^5u\frac{\partial u}{\partial x} \\
0 &= \frac{\partial F_2}{\partial y}(x,y,u(x,y),v(x,y)) = 3y^2v + y^3\frac{\partial v}{\partial y} + 2x^5u\frac{\partial u}{\partial y}
\end{aligned}$$

Therefore,

$$\left.\begin{array}{r} 2 + 6\frac{\partial u}{\partial x}(1,1) + \frac{\partial v}{\partial x}(1,1) = 0 \\ 6\frac{\partial u}{\partial y}(1,1) + \frac{\partial v}{\partial y}(1,1) - 1 = 0 \\ \frac{\partial v}{\partial x}(1,1) + 5 + 2\frac{\partial u}{\partial x}(1,1) = 0 \\ 3 + \frac{\partial v}{\partial y}(1,1) + 2\frac{\partial u}{\partial y}(1,1) = 0 \end{array}\right| \Leftrightarrow \left.\begin{array}{l} 6\frac{\partial u}{\partial x}(1,1) + \frac{\partial v}{\partial x}(1,1) = -2 \\ 2\frac{\partial u}{\partial x}(1,1) + \frac{\partial v}{\partial x}(1,1) = -5 \\ 6\frac{\partial u}{\partial y}(1,1) + \frac{\partial v}{\partial y}(1,1) = 1 \\ 2\frac{\partial u}{\partial y}(1,1) + \frac{\partial v}{\partial y}(1,1) = -3 \end{array}\right| \Leftrightarrow \begin{array}{l} \frac{\partial u}{\partial x}(1,1) = \frac{3}{4}, \\ \frac{\partial v}{\partial x}(1,1) = -\frac{13}{2}, \\ \frac{\partial u}{\partial y}(1,1) = 1, \\ \frac{\partial v}{\partial y}(1,1) = -5. \end{array}$$

7. We have

$$G = \left\{(x,y) \in \mathbb{R}^2 | 2y \le x \le 2,\ 0 \le y \le 1\right\}$$

and

$$\int_0^1 \left(\int_{2y}^2 y\sqrt{1+x^3}dx\right) dy.$$

However, we cannot find an antiderivative with respect to x. Then, we change the order of integration and we have

$$G = \left\{(x,y) \in \mathbb{R}^2 | 0 \le x \le 2,\ 0 \le y \le \frac{x}{2}\right\}$$

and

$$\begin{aligned}
\int\int_G y\sqrt{1+x^3}dxdy &= \int\int_G y\sqrt{1+x^3}dydx = \int_0^2 \left(\int_0^{\frac{x}{2}} y\sqrt{1+x^3}dy\right) dx \\
&= \int_0^2 \left[\frac{1}{2}y^2\sqrt{1+x^3}\right]_0^{\frac{x}{2}} dx = \frac{1}{2}\int_0^2 \frac{x^2}{4}\sqrt{1+x^3}dx \\
&= \frac{1}{24}\int_0^2 (1+x^3)'\sqrt{1+x^3}dx = \frac{1}{24}\left[\frac{2}{3}(1+x^3)^{\frac{3}{2}}\right]_0^2 dx \\
&= \frac{1}{36}(9^{\frac{3}{2}} - 1) = \frac{13}{18}.
\end{aligned}$$

8. We want to calculate $\int\int_G (x^2+y^2)dA$ with

$$G = \left\{(x,y)\in\mathbb{R}^2 | 1\le x+y\le 2,\ 5\le 3x+4y\le 6\right\}.$$

For this, we use the change of coordinates $u = x+y$ and $v = 3x+4y$ and see that $x = 4u-v$ and $y = v-3u$. Then, $\left|\det\begin{pmatrix} \frac{\partial 4u-v}{\partial u} & \frac{\partial 4u-v}{\partial v} \\ \frac{\partial v-3u}{\partial u} & \frac{\partial v-3u}{\partial v}\end{pmatrix}\right| = \left|\det\begin{pmatrix} 4 & -1 \\ -3 & 1\end{pmatrix}\right| = 1 \neq 0$. Then, $dxdy = dudv$, $G_{u,v} = \left\{(u,v)\in\mathbb{R}^2 | 1\le u\le 2,\ 5\le v\le 6\right\}$, and

$$\begin{aligned}
\int\int_G (x^2+y^2)dA &= \int_5^6\int_1^2 ((4u-v)^2+(v-3u)^2)dudv = \int_5^6 \left[\frac{1}{12}(4u-v)^3 - \frac{1}{9}(v-3u)^3\right]_1^2 dv \\
&= \int_5^6 \left(\frac{1}{12}((8-v)^3-(4-v)^3) - \frac{1}{9}((v-6)^3-(v-3)^3)\right)dv \\
&= \left[\frac{1}{48}(-(8-v)^4+(4-v)^4) - \frac{1}{36}((v-6)^4-(v-3)^4)\right]_5^6 \\
&= \frac{1}{48}(-2^4+2^4+3^4-1) - \frac{1}{36}(0-3^4-1+2^4) = \frac{80}{48}+\frac{66}{36} = \frac{7}{2}.
\end{aligned}$$

9. We want to know for which values of $k\in\mathbb{R}$ does the integral $\int\int\int_G \frac{1}{(1+x^2+y^2)^k}dV$ converge with $G = \{(x,y,z)\in\mathbb{R}^3 | z\in[0,1]\}$. For this, we use cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, where $r\ge 0$, $-\pi\le\theta\le\pi$, and $0\le z\le 1$. We already know that $dxdydz = rdrd\theta dz$. Then,

$$\begin{aligned}
\int\int\int_G \frac{1}{(1+x^2+y^2)^k}dV &= \int_0^\infty\int_{-\pi}^\pi\int_0^1 \frac{r}{(1+r^2)^k}dzd\theta dr = \lim_{m\to\infty}\int_0^m\int_{-\pi}^\pi\int_0^1 \frac{r}{(1+r^2)^k}dzd\theta dr \\
&= \lim_{m\to\infty}\int_0^m\int_{-\pi}^\pi \frac{r}{(1+r^2)^k}d\theta dr = \lim_{m\to\infty} 2\pi\int_0^m \frac{r}{(1+r^2)^k}dr \\
&= \lim_{m\to\infty}\pi\int_0^m \frac{(1+r^2)'}{(1+r^2)^k}dr \\
&= \begin{cases} \lim_{m\to\infty}\pi\left[\ln(1+r^2)\right]_0^m & \text{if } k=1, \\ \lim_{m\to\infty}\pi\left[\frac{1}{1-k}(1+r^2)^{1-k}\right]_0^m & \text{if } k\neq 1. \end{cases}
\end{aligned}$$

Then,

$$\int\int\int_G \frac{1}{(1+x^2+y^2)^k}dV = \begin{cases} \lim_{m\to\infty}\pi\ln(1+m^2) = \infty & \text{if } k=1, \\ \lim_{m\to\infty}\pi\frac{1}{1-k}((1+r^2)^{1-k}-1) = \infty & \text{if } k<1, \\ \lim_{m\to\infty}\pi\frac{1}{k-1}(1-\frac{1}{(1+r^2)^{k-1}}) = \frac{\pi}{k-1} & \text{if } k>1. \end{cases}$$

Therefore, our integral only converges for $k>1$ to the value $\frac{\pi}{k-1}$.