1. a) $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0} 1 = 1,$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0}{k^2}-0}{k} = 0.$

b) Consider the limit

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{h^3-hk^2}{h^2+k^2}-0-h-0}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{-2hk^2}{(h^2+k^2)^{\frac{3}{2}}}.$$

This limit is not zero (it does not exist). To see this, we can approach $(0,0)$ via $k=h$, in which case we obtain $\lim_{h\to 0^+} \frac{-2}{\sqrt{8}} \neq 0$.

c) Since the partial derivatives are not continuous at $(0,0)$, we need to use the definition.

$$\frac{\partial f}{\partial \mathbf{v}}(0,0) = \lim_{t\to 0} \frac{f(ta,tb)-f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3a^3-t^3ab^2}{t^2(a^2+b^2)}}{t} = \lim_{t\to 0} \frac{a^3-ab^2}{(a^2+b^2)} = a(a^2-b^2).$$

2. Using the chain rule, we have

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}.$$

Applying the chain rule again and the product rule when needed, we get

$$\begin{aligned}\frac{\partial^2 z}{\partial y\partial x} &= 2x\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v\partial u}\frac{\partial v}{\partial y}\right) + 2\frac{\partial f}{\partial v} + 2y\left(\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)\\ &= 2x\left(-2y\frac{\partial^2 f}{\partial u^2} + 2x\frac{\partial^2 f}{\partial v\partial u}\right) + 2\frac{\partial f}{\partial v} + 2y\left(-2y\frac{\partial^2 f}{\partial u\partial v} + 2x\frac{\partial^2 f}{\partial v^2}\right)\\ &= 4xy\left(\frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial u^2}\right) + 4(x^2-y^2)\frac{\partial^2 f}{\partial v\partial u} + 2\frac{\partial f}{\partial v}.\end{aligned}$$

3. The critical points are given by the equation $Df(x,y) = (0,0)$. In our case,

$$Df(x,y) = \left(\; 4x^3-8(x+y), 4y^3-8(x+y) \;\right) = (0,0)\,.$$

Then,

$$\begin{matrix}4x^3-8(x+y)=0\\4y^3-8(x+y)=0\end{matrix} \;\Leftrightarrow\; \begin{matrix}4x^3-8(x+y)=0\\x^3=y^3\end{matrix} \;\Leftrightarrow\; \begin{matrix}4x^3-16x=0\\y=x\end{matrix} \;\Leftrightarrow\; \begin{matrix}x(x^2-4)=0\\y=x\end{matrix}$$

Thus, we have three critical points: $(0,0)$, $(2,2)$, and $(-2,-2)$. To determine the nature of these points, we need the Hessian matrix.

$$H(x,y) = \begin{pmatrix} 12x^2-8 & -8 \\ -8 & 12y^2-8 \end{pmatrix}.$$

– $(0,0)$:

$$H(0,0) = \begin{pmatrix} -8 & -8 \\ -8 & -8 \end{pmatrix} = A$$

We have $\det(A_1) = -8 < 0$ and $\det(A_2) = 0$. Thus, $H(0,0)$ is negative semidefinite and we need to further investigate the nature of $(0,0)$. Note that for $\delta > 0$, $\left(\frac{\delta}{2}, \frac{\delta}{2}\right), \left(\frac{\delta}{2}, -\frac{\delta}{2}\right) \in B_\delta(0,0)$. For $\delta \in (0,1)$,

$$f\left(\frac{\delta}{2}, \frac{\delta}{2}\right) = \frac{\delta^4}{8} - 4\delta^2 < \frac{\delta^2}{8} - 4\delta^2 = -\frac{31}{8}\delta^2 < 0 = f(0,0)$$

and f does not have a minimum at $(0,0)$. Moreover,

$$f\left(\frac{\delta}{2}, -\frac{\delta}{2}\right) = \frac{\delta^4}{8} > 0 = f(0,0)$$

and f does not have a maximum at $(0,0)$. Therefore, f has a saddle point at $(0,0)$.

– $(2,2)$ and $(-2,-2)$:

$$H(2,2) = H(-2,-2) = \begin{pmatrix} 40 & -8 \\ -8 & 40 \end{pmatrix} = A.$$

We have $\det(A_1) = 40 > 0$ and $\det(A_2) = 1536 > 0$. Thus, $H(2,2)$ (and $H(-2,-2)$) is positive definite and f has a minimum at $(2,2)$ (and at $(-2,-2)$).

4. Our problem is

$$\begin{array}{ll} \min & x^2 + y^2 + z^2 \\ \text{s.t.} & xyz^2 = 2 \end{array}$$

Then, the Lagrange function is $\mathcal{L}(x,y,z,\lambda) = x^2 + y^2 + z^2 + (\lambda xyz^2 - 2)$. The minimum is obtained in one of the critical values of the Lagrange function. We get the system of equations

$$\begin{array}{l} 2x + \lambda yz^2 = 0 \\ 2y + \lambda xz^2 = 0 \\ 2z + 2\lambda xyz = 0 \\ xyz^2 = 2 \end{array} \Leftrightarrow \begin{array}{l} 2x^2 = -\lambda xyz^2 \\ 2y^2 = -\lambda xyz^2 \\ z^2 = -\lambda xyz^2 \\ xyz^2 = 2 \end{array} \Leftrightarrow \begin{array}{l} 2x^2 = -\lambda xyz^2 \\ y^2 = x^2 \\ z^2 = 2x^2 \\ xyz^2 = 2 \end{array} \Leftrightarrow \begin{array}{l} 2x^2 = -\lambda xyz^2 \\ y = \pm x \\ z^2 = 2x^2 \\ xyz^2 = 2 \end{array}$$

For $y = -x$, we get $2 = xyz^2 = x(-x)z^2 = -x^2z^2$ which is not possible. Therefore, $y = x$ and $2 = xyz^2 = x^2(2x^2) = 2x^4$. Hence, $x = \pm 1$, and we have four points :$(1,1,\sqrt{2})$, $(1,1,-\sqrt{2})$, $(-1,-1,\sqrt{2})$, $(-1,-1,-\sqrt{2})$, that give the minimum distance from the origin to the surface $xyz^2 = 2$. This minimal distance is $\sqrt{4} = 2$.

5. a) Let $F(x,y,z) = x^2z + yz^2 - 2$. We want to know if the equation

$$F(x,y,z) = 0$$

has a solution for z as a function of x and y in a neighborhood of $(1,1,-2)$. We know that

- F has continuous partial derivatives,
- $F(1,1,-2) = -2+4-2 = 0$.
- $\frac{\partial F}{\partial z}(1,1,-2) = 1 + 2\cdot 1\cdot(-2) = -3 \neq 0$.

By the implicit function theorem, there is a neighborhood of $(1,1,-2)$ where z can be given as a function of x and y.

b) We have $F(x,y,z(x,y)) = 0$ for all (x,y) in a neighborhood of $(1,1)$. Thus,

$$0 = \frac{\partial F}{\partial x}(x,y,z(x,y)) = 2xz(x,y) + x^2\frac{\partial z}{\partial x}(x,y) + 2yz(x,y)\frac{\partial z}{\partial x}(x,y)$$

and

$$0 = \frac{\partial F}{\partial x}(1,1,-2) = -4 + \frac{\partial z}{\partial x}(1,1) - 4\frac{\partial z}{\partial x}(1,1) = -4 - 3\frac{\partial z}{\partial x}(1,1).$$

Thus, $\frac{\partial z}{\partial x}(1,1) = -\frac{4}{3}$. Next, we have

$$\begin{aligned}0 = \frac{\partial^2 F}{\partial x^2}(x,y,z(x,y)) &= 2z(x,y) + 2x\frac{\partial z}{\partial x}(x,y) + 2x\frac{\partial z}{\partial x}(x,y) + x^2\frac{\partial^2 z}{\partial x^2}(x,y)\\ &\quad + 2y\left(\frac{\partial z}{\partial x}(x,y)\right)^2 + 2yz(x,y)\frac{\partial^2 z}{\partial x^2}(x,y)\end{aligned}$$

and

$$0 = \frac{\partial^2 F}{\partial x^2}(1,1,-2) = -4 + 4\left(-\frac{4}{3}\right) + \frac{\partial^2 z}{\partial x^2}(1,1) + 2\left(-\frac{4}{3}\right)^2 - 4\frac{\partial^2 z}{\partial x^2}(1,1).$$

Thus, $\frac{\partial^2 z}{\partial x^2}(1,1) = -\frac{52}{27}$.

6. We have

$$G = \{(x,y) \in \mathbb{R}^2 | \frac{3}{2}y \leq x \leq 3, 0 \leq y \leq 2\}$$

and

$$\int\int_G y^2 \sin x^4 dxdy = \int_0^2 dy \int_{\frac{3}{2}y}^3 y^2 \sin x^4 dx$$

which we do not know how to solve. Then, we change the order of integration and we have

$$G = \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 3, 0 \leq y \leq \frac{2}{3}x\}$$

and

$$\begin{aligned}\int\int_G y^2 \sin x^4 dxdy = \int\int_G y^2 \sin x^4 dydx = \int_0^3 dx \int_0^{\frac{2}{3}x} y^2 \sin x^4 dy = \int_0^3 \frac{1}{3}\left(\frac{2}{3}x\right)^3 \sin x^4 dx\\ = \frac{8}{81}\int_0^3 x^3 \sin x^4 dx = \frac{2}{81}\int_0^3 (x^4)' \sin x^4 dx = \frac{2}{81}\left[-\cos x^4\right]_0^3\\ = \frac{2}{81}(1-\cos 81).\end{aligned}$$

7. We use the substitution $u = x^2 + y^2$, $v = \frac{y}{x}$. Then,

$$\left|\det\begin{pmatrix} 2x & 2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}\right| = 2\left(1 + \frac{y^2}{x^2}\right) \neq 0.$$

Hence, $dudv = 2\left(1 + \frac{y^2}{x^2}\right) dA$, $1 \leq u \leq 9$, and $0 \leq v \leq \frac{1}{4}$. Thus,

$$\int\int_G \frac{y}{x} dA = \int\int_G \frac{y}{x} \frac{2\left(1+\frac{y^2}{x^2}\right)}{2\left(1+\frac{y^2}{x^2}\right)} dA = \int_0^{\frac{1}{4}} dv \int_1^9 \frac{v}{2(1+v^2)} du$$

$$= 8\int_0^{\frac{1}{4}} \frac{v}{2(1+v^2)} dv = 2\int_0^{\frac{1}{4}} \frac{2v}{1+v^2} dv = 2\left[\ln(1+v^2)\right]_0^{\frac{1}{4}} = 2\ln\left(\frac{17}{16}\right) = \ln\left(\frac{17}{16}\right)^2.$$

8. Spherical coordinates are given by $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$, where $r \geq 0$, $0 \leq \theta \leq \pi$, and $-\pi \leq \varphi \leq \pi$; further, $dV = r^2 \sin\theta dr d\theta d\varphi$. Let

$$G_m = \{(x, y, z) \in \mathbb{R}^2 \mid \frac{1}{m^2} \leq x^2 + y^2 + z^2 \leq m^2\}.$$

Then,

$$\int\int\int_{\mathbb{R}^3} \frac{e^{k(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}} dV = \lim_{m\to\infty} \int\int\int_{G_m} \frac{e^{k(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}} dV$$

$$= \lim_{m\to\infty} \int_{\frac{1}{m}}^m dr \int_0^\pi d\theta \int_{-\pi}^\pi \frac{e^{kr^2}}{r} r^2 \sin\theta d\varphi$$

$$= \lim_{m\to\infty} \int_{\frac{1}{m}}^m dr \int_0^\pi 2\pi r e^{kr^2} \sin\theta d\theta$$

$$= \lim_{m\to\infty} \int_{\frac{1}{m}}^m 2\pi r e^{kr^2} \left[-\cos\theta\right]_0^\pi dr = \lim_{m\to\infty} 4\pi \int_{\frac{1}{m}}^m r e^{kr^2} dr.$$

For $k = 0$, we have

$$\lim_{m\to\infty} 4\pi \int_{\frac{1}{m}}^m r dr = \lim_{m\to\infty} 2\pi\left(m^2 - \frac{1}{m^2}\right) = \infty$$

and the integral does not converge. For $k \neq 0$, we have

$$\lim_{m\to\infty} 4\pi \int_{\frac{1}{m}}^m r e^{kr^2} dr = \lim_{m\to\infty} 4\pi \left[\frac{e^{kr^2}}{2k}\right]_{\frac{1}{m}}^m = \lim_{m\to\infty} \frac{2\pi}{k}\left(e^{km^2} - e^{\frac{k}{m^2}}\right).$$

For $k > 0$,

$$\lim_{m\to\infty} \frac{2\pi}{k}\left(e^{km^2} - e^{\frac{k}{m^2}}\right) = \frac{2\pi}{k}\left(e^{\infty} - e^0\right) = \infty$$

and the integral does not converge. For $k < 0$,

$$\lim_{m\to\infty} \frac{2\pi}{k}\left(e^{km^2} - e^{\frac{k}{m^2}}\right) = \frac{2\pi}{k}\left(e^{-\infty} - e^0\right) = -\frac{2\pi}{k}$$

and the integral converges.