# Exam Advanced Logic

VU University Amsterdam, 1 June 2015, 18:30–21:15

This exam consists of four questions. In total you can score 90 points as indicated per question. The final grade is the minimum of 10 and (points/10 + 1 + bonus).

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**1.** **(a)** Show that the formula $\Box p \to \Diamond\Box\Diamond p$ is valid on all reflexive frames. *(8 pt)*

**(b)** Is the formula $\Box p \to \Diamond\Box\Diamond p$ valid *only* on reflexive frames? Motivate your answer. *(8 pt)*

**(c)** What frame property is characterised by the formula $\Box p \vee \Box\neg p$? Give a formal definition of the property, and prove the characterisation. *(8 pt)*

**2.** Consider the frames $\mathcal{A}$ and $\mathcal{B}$ defined for the language with modal operators $\langle a \rangle$ and $\langle b \rangle$ defined by $\mathcal{A} = (W^{\mathcal{A}}, R_a^{\mathcal{A}}, R_b^{\mathcal{A}})$ and $\mathcal{B} = (W^{\mathcal{B}}, R_a^{\mathcal{B}}, R_b^{\mathcal{B}})$, where

$$
\begin{aligned}
W^{\mathcal{A}} &= \{s, t\} & W^{\mathcal{B}} &= \mathbb{N} = \{0, 1, 2, \ldots\} \\
R_a^{\mathcal{A}} &= \{(s, t)\} & R_a^{\mathcal{B}} &= \{(2n, 2n+1) \mid n \in \mathbb{N}\} \\
R_b^{\mathcal{A}} &= \{(t, s)\} & R_b^{\mathcal{B}} &= \{(2n+1, 2n+2) \mid n \in \mathbb{N}\}\,.
\end{aligned}
$$

**(a)** If possible, give a formula valid in $\mathcal{A}$ but not in $\mathcal{B}$, and prove both facts. Otherwise, explain why such a formula does not exist. *(8 pt)*

**(b)** Same question as **2 (a)** but now with the roles of $\mathcal{A}$ and $\mathcal{B}$ interchanged. *(8 pt)*

Consider the valuations $V^{\mathcal{A}}$ and $V^{\mathcal{B}}$ defined on the respective frames $\mathcal{A}$ and $\mathcal{B}$ by

$$
V^{\mathcal{A}}(p) = \{s\} \qquad\qquad V^{\mathcal{B}}(p) = \{2n \mid n \in \mathbb{N}\}\,.
$$

**(c)** Show that state $s$ of the model $(\mathcal{A}, V^{\mathcal{A}})$ is bisimilar to state 42 of the model $(\mathcal{B}, V^{\mathcal{B}})$. *(8 pt)*

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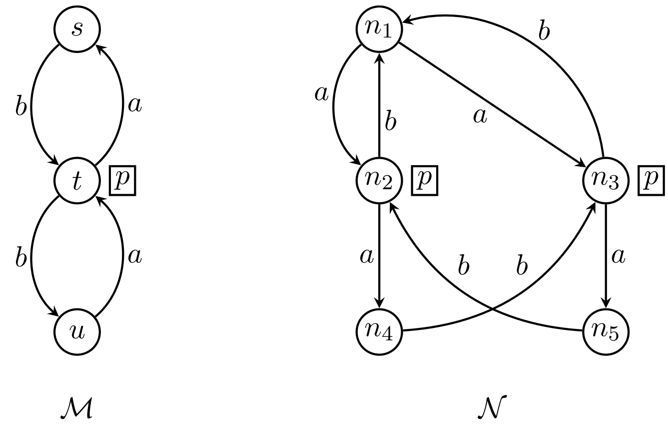
**3.** Consider the models $\mathcal{M}$ and $\mathcal{N}$ defined by:




**(a)** Show that there is no modal formula distinguishing state $n_3$ in model $\mathcal{N}$ from state $t$ in model $\mathcal{M}$. *(8 pt)*

**(b)** Let $\widehat{\mathcal{N}}$ be the PDL-extension of model $\mathcal{N}$. Compute the transition relation $\widehat{R}_\beta$ corresponding to the PDL-program $\beta = \mathsf{while}\ p\ \mathsf{do}\ abba$. *(8 pt)*

**(c)** Determine whether the PDL-formula $[\beta]p \leftrightarrow p$ globally holds in $\widehat{\mathcal{N}}$. Prove your answer. *(6 pt)*

**4.** The Hilbert system for the logic $S5$ is the extension of the Hilbert system for the basic modal logic $K$ with $A1$: the truth axiom (if something is known, it is true), $A2$: the axiom of positive introspection, and $A3$: the axiom of negative introspection.

**(a)** Prove that every reflexive and Euclidean relation is transitive. How can this be used to show that $A2$ follows from $A1$ and $A3$? *(7 pt)*

**(b)** Formulate the completeness theorem for $S5$. *(5 pt)*

**(c)** Show that $\neg K \neg(p \wedge Kq) \leftrightarrow (\neg K \neg p \wedge Kq)$ is a theorem of $S5$. (For this you may use your answer to **4 (b)**.) *(8 pt)*