Exam Advanced Logic

VU University Amsterdam, 25 March 2015, 15:15–18:00

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed. In total you can score 90 points as indicated per question. The final grade is the minimum of 10 and (points/10 + 1 + bonus).

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1. **(a)** What frame property is characterized by the formula $p \leftrightarrow \Box p$? Prove your answer. *(10 pt)*

(b) Show that $q \leftrightarrow \Diamond q$ characterizes the same frame property. (Note that for this you do not have to know the frame property asked for in **(a)**.) *(10 pt)*

2. **(a)** Show that the formula $\Box\Diamond p \rightarrow \Box\Diamond\Box\Diamond p$ is valid on all transitive frames. *(10 pt)*

(b) Is the formula $\Box\Diamond p \rightarrow \Box\Diamond\Box\Diamond p$ *only* valid on transitive frames? Prove your answer. *(10 pt)*

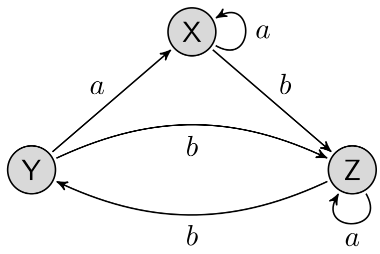
3. System *S5* is the extension of system K with the truth axiom (if something is known, it is true), the axiom of positive introspection, and the axiom of negative introspection.

(a) Show that $p \rightarrow K\neg K\neg p$ is a theorem of *S5*. *(10 pt)*

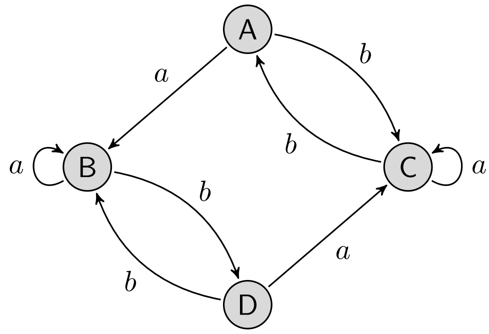
(b) Show that the following rule is admissible in *S5*:

$$\frac{\varphi \rightarrow K\psi}{\neg K\neg\varphi \rightarrow \psi}$$ *(10 pt)*

4. Consider the following $\{a, b\}$-frames $\mathcal{F}_1$ and $\mathcal{F}_2$:



$\mathcal{F}_1$



$\mathcal{F}_2$

(a) Explain why in all models on these frames, the meanings of $\langle i\rangle\varphi$ and $[i]\varphi$ coincide for $i \in \{a, b\}$. That is, explain why in all models $\mathcal{M}$ on a frame $\mathcal{F} \in \{\mathcal{F}_1, \mathcal{F}_2\}$ for each label $i \in \{a, b\}$, for all states s, and for all formulas φ it holds: $\mathcal{M}, s \vDash [i]\varphi \iff \mathcal{M}, s \vDash \langle i\rangle\varphi$. (An informal proof suffices.) *(5 pt)*

(b) Draw the first three levels of the tree unravelling (or unfolding) of $\mathcal{F}_2$ taking state A as root node. *(5 pt)*

Now consider the models $\mathcal{M}_1 = (\mathcal{F}_1, V_1)$ and $\mathcal{M}_2 = (\mathcal{F}_2, V_2)$ where $V_1(p) = \{\mathsf{Z}\}$ and $V_2(p) = \{\mathsf{C}, \mathsf{D}\}$, and $V_1(q) = V_2(q) = \varnothing$ for all propositional variables $q \neq p$.

(c) If possible, give a multi-modal formula over the index set $\{a, b\}$ that distinguishes state Y of $\mathcal{M}_1$ from state A of $\mathcal{M}_2$. Otherwise, prove that there is no such formula. *(10 pt)*

The following question is about propositional dynamic logic (PDL).

(d) Prove that $p \leftrightarrow [a^*(bb)^*]p$ is globally true in the PDL-extension of $\mathcal{M}_1$. *(10 pt)*