Exam Advanced Logic

Vrije Universiteit Amsterdam, 2 June 2014, 18:30–21:15

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed. In total you can score 90 points as indicated per question. The final grade is the minimum of 10 and (points/10 + 1 + bonus).

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1. **(a)** Prove or disprove the validity of $\Box\Diamond\Box p \to \Diamond p$ in reflexive frames. *(5 pt)*

(b) Prove or disprove the validity of $p \to \Diamond\Box p$ in reflexive, transitive frames. *(5 pt)*

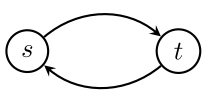
(c) Is the formula $\Box p \to \Box\Box\Box p$ valid in transitive frames? Is it valid only in transitive frames? Prove your answers. *(7 pt)*

(d) Show that the frame property $\forall xy\,(Rxy \to x = y)$ is characterized by the modal formula $p \to \Box p$. *(7 pt)*

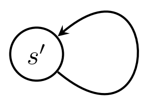
2. Consider the extension of the basic modal language with the 'difference' operator D whose semantics is given by:

$$\mathcal{M}, s \vDash D\varphi \quad \text{if and only if} \quad \text{there exists } t \neq s \text{ such that } \mathcal{M}, t \vDash \varphi\,.$$

Let $\mathcal{F}$ en $\mathcal{G}$ be the frames defined by the following pictures:



$\mathcal{F}$



$\mathcal{G}$

(a) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{F}$? Explain your answer. *(5 pt)*

(b) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{G}$? Explain your answer. *(5 pt)*

(c) Show that the formula $\Diamond p \to Dp$ characterizes irreflexivity. *(7 pt)*

(d) Show that D is not definable in the basic modal language. *(7 pt)*

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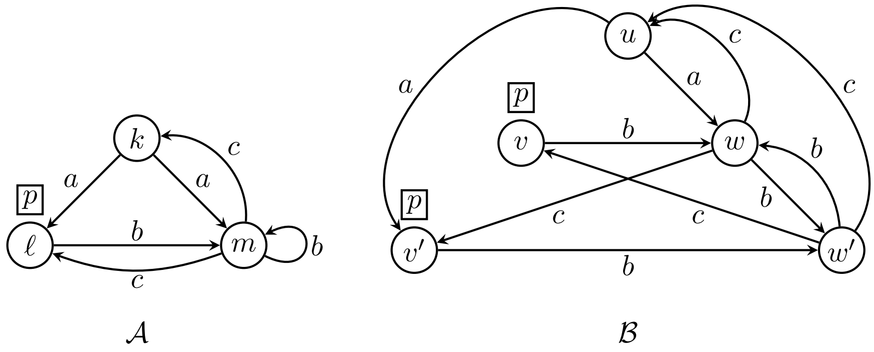
3. Consider the $\{a, b, c\}$-models $\mathcal{A}$ and $\mathcal{B}$ defined by:



(a) Define a relation $E \subseteq \{k, \ell, m\} \times \{u, v, w, v', w'\}$ such that $E : \mathcal{A}, k \leftrightarroweq \mathcal{B}, u$. Show that E satisfies the forward condition (*zig*) of bisimulations. *(8 pt)*

(b) Is there a formula which is true in state m but false in state w? Explain. *(4 pt)*

(c) Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\pi$ corresponding to the PDL-program π:

$$\pi = \textsf{if } p \textsf{ then } (bc)^* \textsf{ else } (a \cup c)^* \qquad (8\ pt)$$

(d) Determine for each of the PDL-formulae $\langle \pi \rangle p$ and $[\pi]p$ whether it holds in $\widehat{\mathcal{A}}$. *(4 pt)*

4. The Hilbert system T is the extension of the system K with the axiom of veridicality $A1$ (if something is known, it is true). System $S4$ is the extension of T with $A2$, the axiom of positive introspection. System $S5$ is the extension of $S4$ with $A3$, the axiom of negative introspection.

(a) Formulate the completeness theorem for the system $S5$. *(3 pt)*

(b) Give a derivation or construct a countermodel:

(i) $\vdash_T (K(p \to q) \wedge p) \to q$ *(5 pt)*

(ii) $\vdash_{S4} \neg KKp \to K\neg Kp$ *(5 pt)*

(iii) $\vdash_{S5} \neg KKp \to K\neg Kp$ *(5 pt)*