

Possible Answers to Exam Advanced Logic

VU University Amsterdam, 26 March 2014, 15:15–18:00

Dimitri Hendriks

1. (a) Define what it means that a modal formula is globally true in a model. (3 pt)

A formula φ is globally true in a model $\mathcal{M} = (W, R, V)$, which we denote by $\mathcal{M} \models \varphi$, if φ is true in all of its points, that is,

$$\mathcal{M} \models \varphi \iff \mathcal{M}, w \models \varphi, \text{ for all } w \in W.$$

- (b) Define what it means that a modal formula is valid in a frame. (3 pt)

A formula φ is valid in a frame $\mathcal{F} = (W, R)$, which we denote by $\mathcal{F} \models \varphi$, if φ is globally true in all models based on \mathcal{F} , that is,

$$\mathcal{F} \models \varphi \iff (\mathcal{F}, V) \models \varphi, \text{ for all valuations } V : \Omega \rightarrow \mathbf{2}^W,$$

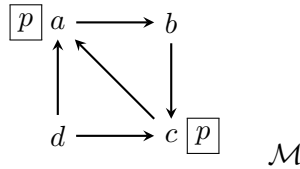
where Ω is a set of propositional variables containing the variables that occur in φ .

Consider the frame $\mathcal{F} = (W, R)$ with W and R given by

$$W = \{a, b, c, d\} \quad R = \{(a, b), (b, c), (c, a), (d, a), (d, c)\}$$

and the model $\mathcal{M} = (\mathcal{F}, V)$ with valuation V defined by $V(p) = \{a, c\}$.

- (c) Give a graphical representation of \mathcal{M} . (2 pt)

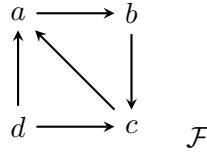


(d) Prove that $p \rightarrow \Box\Box\Box p$ is globally true in \mathcal{M} , but not valid in \mathcal{F} . (4+4 pt)

- To prove that $\mathcal{M} \models p \rightarrow \Box\Box\Box p$, we have to show $\mathcal{M}, x \models p \rightarrow \Box\Box\Box p$ for all $x \in \{a, b, c, d\}$. In points b and d the implication $p \rightarrow \Box\Box\Box p$ is trivially true because they do not satisfy p . To see that $\mathcal{M}, a \models \Box\Box\Box p$ we note that¹ $R^3[a] = \{a\}$ and $\mathcal{M}, a \models p$ (a is the only third R -successor of a), or, step by step: $\mathcal{M}, c \models \Box p$ (by $R[c] = \{a\}$ and $\mathcal{M}, a \models p$), hence $\mathcal{M}, b \models \Box\Box p$ (by $R[b] = \{c\}$), and hence $\mathcal{M}, a \models \Box\Box\Box p$ (by $R[a] = \{b\}$). Likewise we see that $\mathcal{M}, c \models \Box\Box\Box p$ since also $R^3[c] = \{c\}$ and $\mathcal{M}, c \models p$. So both a and c satisfy $\Box\Box\Box p$, and so they satisfy $p \rightarrow \Box\Box\Box p$. Thus we have seen that $\mathcal{M}, x \models p \rightarrow \Box\Box\Box p$ for all points x of the model, and we conclude $\mathcal{M} \models p \rightarrow \Box\Box\Box p$.
- To prove that $\mathcal{F} \not\models p \rightarrow \Box\Box\Box p$, we have to find a valuation V' on \mathcal{F} and a point $x \in W$ such that $\mathcal{F}, V', x \not\models p \rightarrow \Box\Box\Box p$.

We take $V'(p) = \{d\}$. Then, clearly $\mathcal{F}, V', d \models p$ and $\mathcal{F}, V', d \not\models \Box\Box\Box p$ because $b \in R^3[d]$ and $\mathcal{F}, V', b \not\models p$.

(e) Prove that for any formula φ , the formula $\Box\varphi \leftrightarrow \Box\Box\Box\Box\varphi$ is valid in \mathcal{F} . (8 pt)



This follows from the observation that, in the frame \mathcal{F} , $R^4 = R$, and the general fact that $\mathcal{G} \models [\alpha]p \leftrightarrow [\beta]p$ follows from $R_\beta = R_\alpha$, for all frames \mathcal{G} with relations R_α and R_β . A more ad hoc proof goes as follows:

We show that $\mathcal{F} \models \Box p \leftrightarrow \Box\Box\Box\Box p$. Then $\mathcal{F} \models \Box\varphi \leftrightarrow \Box\Box\Box\Box\varphi$, with φ an arbitrary formula, follows since validity is closed under substitution, that is, for all frames \mathcal{G} , modal formulas ψ and substitutions σ , if $\mathcal{G} \models \psi$ then $\mathcal{G} \models \psi^\sigma$.

For $x \in \{a, b, c\}$ we write x' to denote the (unique) y such that Rxy ; so $a' = b$, $b' = c$, and $c' = a$; clearly we have $x''' = x$. For $x \in \{a, b, c\}$ we see that

$$\mathcal{N}, x \models \Box\Box\Box\Box p \iff \mathcal{N}, x' \models \Box\Box\Box p$$

¹Recall that we use the notation $R[x]$ for the set of R -successors of x , i.e., $R[x] = \{y \mid Rxy\}$. Moreover $R^3 = R; R; R$ where $R; S$ denotes the relational composition of R and S , that is, $R; S = \{(x, y) \mid \exists u ((x, u) \in R \text{ and } (u, y) \in S)\}$. By the way, instead of the semi-colon ‘;’, many people use the symbol \circ .

$$\begin{aligned}
&\Longleftrightarrow \mathcal{N}, x'' \models \Box\Box p \\
&\Longleftrightarrow \mathcal{N}, x''' \models \Box p \\
&\Longleftrightarrow \mathcal{N}, x \models \Box p \\
&\Longleftrightarrow \mathcal{N}, x' \models p
\end{aligned}$$

For d we find

$$\begin{aligned}
\mathcal{N}, d \models \Box\Box\Box\Box p &\Longleftrightarrow \mathcal{N}, a \models \Box\Box\Box p \text{ and } \mathcal{N}, c \models \Box\Box\Box p \\
&\Longleftrightarrow \mathcal{N}, a \models p \text{ and } \mathcal{N}, c \models p \\
&\Longleftrightarrow \mathcal{N}, d \models \Box p
\end{aligned}$$

Thus we have shown that $\mathcal{F}, U, x \models \Box\Box\Box\Box p$ iff $\mathcal{F}, U, x \models \Box p$ for all $x \in \{a, b, c, d\}$, i.e., that $\mathcal{F}, U \models \Box p \leftrightarrow \Box\Box\Box\Box p$.

Since U was an arbitrary valuation, we conclude that $\mathcal{F} \models \Box p \leftrightarrow \Box\Box\Box\Box p$.

2. (a) Let I be an arbitrary index set, and let $i, j \in I$. Prove that the formula $p \rightarrow [i]\langle j \rangle p$ characterizes the class of I -frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ that satisfy the property $R_i \subseteq R_j^{-1}$. (10 pt)

The multi-modal formula $p \rightarrow [i]\langle j \rangle p$ characterizes the property $R_i \subseteq R_j^{-1}$ if

$$\mathcal{F} \models p \rightarrow [i]\langle j \rangle p \Longleftrightarrow R_i \subseteq R_j^{-1}$$

for all I -frames $\mathcal{F} = (W, \{R_k \mid k \in I\})$ and $i, j \in I$. So let \mathcal{F} be an arbitrary I -frame, and let $i, j \in I$. We prove the two directions:

- (\Rightarrow) By contraposition. Assume that $R_i \not\subseteq R_j^{-1}$. We prove $\mathcal{F} \not\models p \rightarrow [i]\langle j \rangle p$. By the assumption there are (not necessarily distinct) points a and b such that $R_i a b$ and $\neg R_j b a$. In order to show that $p \rightarrow [i]\langle j \rangle p$ is not valid in \mathcal{F} we have to find a valuation V on \mathcal{F} and a point x such that $\mathcal{M}, x \models p$ and $\mathcal{M}, x \not\models [i]\langle j \rangle p$.

We choose V to be such that p holds in a only, $V(p) = \{a\}$. Then in the model $\mathcal{M} = (\mathcal{F}, V)$ we have $\mathcal{M}, b \not\models \langle j \rangle p$ since $\neg R_j b a$. Hence, due to $R_i a b$, also $\mathcal{M}, a \not\models [i]\langle j \rangle p$. We conclude that $\mathcal{M}, a \not\models p \rightarrow [i]\langle j \rangle p$. Hence $\mathcal{F} \not\models p \rightarrow [i]\langle j \rangle p$.

- (\Leftarrow) Assume $R_i \subseteq R_j^{-1}$, that is, $R_i u v$ implies $R_j v u$, for all points u and v .

We have to show that $p \rightarrow [i]\langle j \rangle p$ is valid in \mathcal{F} . Let V be an arbitrary valuation on \mathcal{F} , x an arbitrary point in the model $\mathcal{M} = (\mathcal{F}, V)$, and assume $\mathcal{M}, x \models p$. In order to show $\mathcal{M}, x \models [i]\langle j \rangle p$, we consider an arbitrary R_i -successor y of x , $R_i xy$, and prove $\mathcal{M}, y \models \langle j \rangle p$. By the assumption $R_i \subseteq R_j^{-1}$ we know that $R_j yx$. Hence, since we have $\mathcal{M}, x \models p$, it follows that $\mathcal{M}, y \models \langle j \rangle p$.

- (b) Use the result of the previous question to show that the formula $\langle i \rangle [j] p \rightarrow p$ also characterizes the frame property $R_i \subseteq R_j^{-1}$. (7 pt)

We reason as follows

$$\mathcal{F} \models \langle i \rangle [j] p \rightarrow p \iff \mathcal{F} \models \langle i \rangle [j] \neg p \rightarrow \neg p \quad (1)$$

$$\iff \mathcal{F} \models p \rightarrow \neg \langle i \rangle [j] \neg p \quad (2)$$

$$\iff \mathcal{F} \models p \rightarrow [i] \neg [j] \neg p \quad (3)$$

$$\iff \mathcal{F} \models p \rightarrow [i] \langle j \rangle p \quad (4)$$

$$\iff R_i \subseteq R_j^{-1} \quad (5)$$

where the steps are justified as follows:

- (1) The direction \Rightarrow follows from the fact that validity is closed under substitution; here we substitute $\neg p$ for p . The direction \Leftarrow uses additionally that we may replace subformulas by equivalent subformulas; so from $\mathcal{F} \models \langle i \rangle [j] \neg p \rightarrow \neg p$ we infer $\mathcal{F} \models \langle i \rangle [j] \neg \neg p \rightarrow \neg \neg p$ and then replace $\neg \neg p$ by p .
 - (2) These are equivalent since one formula is the contraposition of the other.
 - (3) $\neg \langle i \rangle [j] \neg p$ is equivalent to $[i] \neg [j] \neg p$.
 - (4) $\neg [j] \neg p$ is equivalent to $\langle j \rangle p$.
 - (5) By the result proven in 2.(a).
- (c) Are the formulas $p \rightarrow [i]\langle j \rangle p$ and $\langle i \rangle [j] p \rightarrow p$ equivalent? Prove your answer. (8 pt)

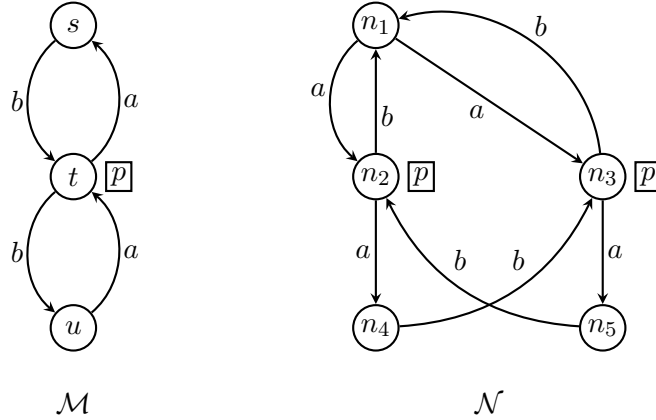
No, they are not. Clearly, *inside* the class of I -frames with the property $R_i \subseteq R_j^{-1}$ they are equivalent, as we just have shown that they are both valid in that class. However, *outside* this class they need not be equivalent, as we show by the following counterexample.

First we recall the definition of equivalence of modal formulas: Two formulas φ and ψ are *equivalent*, which we denote by $\varphi \equiv \psi$, if $\varphi \leftrightarrow \psi$ is universally

valid. In other words, $\varphi \equiv \psi$ when $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}, x \models \psi$ for all models \mathcal{M} and all points x of \mathcal{M} .

Now consider the following model $\mathcal{M} = (\{a\}, \{R_1, R_2\}, V)$ with $R_1 = \{(a, a)\}$, $R_2 = \emptyset$, and $V(p) = \emptyset$. Then we have $\mathcal{M}, a \models p \rightarrow [1]\langle 2 \rangle p$ because of $\mathcal{M}, a \not\models p$. On the other hand $\mathcal{M}, a \models [2]p$ by $R_2 = \emptyset$, and so, by $R_1 aa$, we have $\mathcal{M}, a \models \langle 1 \rangle [2]p$. In combination with $\mathcal{M}, a \not\models p$ this gives $\mathcal{M}, a \not\models \langle 1 \rangle [2]p \rightarrow p$. We conclude that $(p \rightarrow [1]\langle 2 \rangle p) \not\equiv (\langle 1 \rangle [2]p \rightarrow p)$.

3. Consider the $\{a, b\}$ -models \mathcal{M} and \mathcal{N} defined by:



(a) Define model \mathcal{M} by means of set notation.

(2 pt)

$$\begin{aligned}\mathcal{M} &= (W^{\mathcal{M}}, R_a^{\mathcal{M}}, R_b^{\mathcal{M}}, V^{\mathcal{M}}) \\ W^{\mathcal{M}} &= \{s, t, u\} \\ R_a^{\mathcal{M}} &= \{(t, s), (u, t)\} \\ R_b^{\mathcal{M}} &= \{(s, t), (t, u)\} \\ V^{\mathcal{M}}(p) &= \{t\}\end{aligned}$$

(Likewise we will use $\mathcal{N} = (W^{\mathcal{N}}, R_a^{\mathcal{N}}, R_b^{\mathcal{N}}, V^{\mathcal{N}})$.)

(b) Is there a modal formula that distinguishes state n_3 in model \mathcal{N} from state t in model \mathcal{M} ? Prove your answer. (10 pt)

No, there is no such formula. We show that t and n_3 are bisimilar, and we know that bisimilar states have the same modal theory: if pointed models

\mathcal{X}, x and \mathcal{X}', x' are bisimilar then, for all modal formulas φ , it holds that $\mathcal{X}, x \models \varphi$ if and only if $\mathcal{X}', x' \models \varphi$.

Define the relation $G \subseteq W^{\mathcal{M}} \times W^{\mathcal{N}}$ by

$$G := \{(s, n_4), (s, n_5), (t, n_2), (t, n_3), (u, n_1)\}.$$

We show that G is a bisimulation:

- First of all, we notice that G satisfies the requirement of atomic harmony: for all $(x, x') \in G$ and all propositional variables q we have $\mathcal{M}, x \models q$ iff $\mathcal{N}, x' \models q$.
- To verify the zig-condition of G , for every pair $(x, x') \in G$, for every $i \in \{a, b\}$, and for every $y \in W^{\mathcal{M}}$ with $R_i^{\mathcal{M}}xy$, we have to find a point $y' \in W^{\mathcal{N}}$ such that $R_i^{\mathcal{N}}x'y'$ and $(y, y') \in G$. This we indicate by $\frac{x}{y} \Big| \frac{x'}{y'} i$.

$$\frac{s}{t} \Big| \frac{n_4}{n_3} b \quad \frac{s}{t} \Big| \frac{n_5}{n_2} b \quad \frac{t}{s} \Big| \frac{n_2}{n_4} a \quad \frac{t}{u} \Big| \frac{n_2}{n_1} b \quad \frac{t}{s} \Big| \frac{n_3}{n_5} a \quad \frac{t}{u} \Big| \frac{n_3}{n_1} b \quad \frac{u}{t} \Big| \frac{n_1}{n_3} a$$

- Similarly for diagrams showing the zag condition (when a step $R_i^{\mathcal{N}}x'y'$ has to be matched by a step $R_i^{\mathcal{M}}xy$) we write $\frac{x}{y} \Big| \frac{x'}{y'} i$.

$$\frac{s}{t} \Big| \frac{n_4}{n_3} b \quad \frac{s}{t} \Big| \frac{n_5}{n_2} b \quad \frac{s}{t} \Big| \frac{n_5}{n_2} b \quad \frac{t}{s} \Big| \frac{n_2}{n_4} a \quad \frac{t}{u} \Big| \frac{n_2}{n_1} b \quad \frac{t}{s} \Big| \frac{n_3}{n_5} a \quad \frac{t}{u} \Big| \frac{n_3}{n_1} b \quad \frac{u}{t} \Big| \frac{n_1}{n_2} a \quad \frac{u}{t} \Big| \frac{n_1}{n_3} a$$

(One diagram more than for zig due to two outgoing a -steps from n_1 .)

- (c) Let $\hat{\mathcal{N}}$ be the PDL-extension of model \mathcal{N} . Compute the transition relation \hat{R}_π corresponding to the PDL-program $\pi = \text{if } p \text{ then } ba \text{ else } ab$. (8 pt)

In PDL syntax we have $(\text{if } p \text{ then } ba \text{ else } ab) = (p?; ba) \cup (\neg p?; ab)$. We compute the transition relations of the component programs:

$$\begin{aligned} \hat{R}_a &= R_a = \{(n_1, n_2), (n_1, n_3), (n_2, n_4), (n_3, n_5)\} \\ \hat{R}_b &= R_b = \{(n_2, n_1), (n_3, n_1), (n_4, n_3), (n_5, n_2)\} \\ \hat{R}_{ba} &= \hat{R}_b; \hat{R}_a = \{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3), (n_4, n_5), (n_5, n_4)\} \\ \hat{R}_{p?} &= \{(x, x) \mid \mathcal{N}, x \models p\} = \{(n_2, n_2), (n_3, n_3)\} \\ \hat{R}_{p?ba} &= \hat{R}_{p?}; \hat{R}_{ba} = \{(n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3)\} \end{aligned}$$

$$\begin{aligned}
\widehat{R}_{ab} &= \widehat{R}_a; \widehat{R}_b = \{(n_1, n_1), (n_2, n_3), (n_3, n_2)\} \\
\widehat{R}_{\neg p?} &= \{(x, x) \mid \mathcal{N}, x \models \neg p\} = \{(n_1, n_1), (n_4, n_4), (n_5, n_5)\} \\
\widehat{R}_{\neg p?ab} &= \widehat{R}_{\neg p?}; \widehat{R}_{ab} = \{(n_1, n_1)\} \\
\widehat{R}_\pi &= \widehat{R}_{p?ba} \cup \widehat{R}_{\neg p?ab} = \{(n_1, n_1), (n_2, n_2), (n_2, n_3), (n_3, n_2), (n_3, n_3)\}
\end{aligned}$$

- (d) Determine whether the PDL-formula $[b]\perp \rightarrow ([\pi]p \rightarrow \perp)$ globally holds in $\widehat{\mathcal{N}}$. Prove your answer. (6 pt)

Yes, $\widehat{\mathcal{N}} \models [b]\perp \rightarrow ([\pi]p \rightarrow \perp)$ holds. To see this, we only have to consider the point n_1 , as this is the only point that is blind with respect to the relation R_b ; in all other points $x \neq n_1$ we have $\mathcal{N}, x \not\models [b]\perp$ and so the implication $[b]\perp \rightarrow ([\pi]p \rightarrow \perp)$ is trivially true there. So $\widehat{\mathcal{N}}, n_1 \models [b]\perp$. Now for $\widehat{\mathcal{N}}, n_1 \models [\pi]p \rightarrow \perp$ to hold we have to verify that $\widehat{\mathcal{N}}, n_1 \not\models [\pi]p$, i.e., we need an R_π -successor of n_1 where p does not hold. Indeed, we have $(n_1, n_1) \in R_\pi$ and $\widehat{\mathcal{N}}, n_1 \not\models p$ and so $\widehat{\mathcal{N}}, n_1 \not\models [\pi]p$.

4. System T is the extension of the minimal modal logic K with the axiom of veridicality (if something is known, it is true). System S_4 extends T with the axiom of positive introspection; S_5 extends S_4 with the axiom of negative introspection.

Prove or disprove the following epistemic claims (you may use completeness theorems):

- (a) $\vdash_T p \rightarrow \neg K \neg p$ (5 pt)

1. $Kp \rightarrow p$ (veridicality)
2. $K \neg p \rightarrow \neg p$ (subst. instance of 1)
3. $(a \rightarrow \neg b) \rightarrow (b \rightarrow \neg a)$ (tautology)
4. $(K \neg p \rightarrow \neg p) \rightarrow (p \rightarrow \neg K \neg p)$ (subst. instance of 3)
5. $p \rightarrow \neg K \neg p$ (modus ponens, 4, 2)

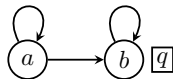
- (b) $\vdash_{S_4} q \vee K \neg K q$ (5 pt)

We show that $q \vee K \neg K q$ cannot be derived in S_4 by applying the completeness theorem for S_4 which states

$$\vdash_{S_4} \varphi \iff \text{Refl} \cap \text{Trans} \models \varphi.$$

That is, we give a concrete model $\mathcal{M} = (W, R, V)$ with R reflexive and transitive, and a point x such that both $\mathcal{M}, x \not\models q$ and $\mathcal{M}, x \not\models K \neg K q$.

Consider the following model based on a reflexive and transitive frame:



In this model we have that $a \not\models q$ and $a \not\models K\neg Kq$. The latter holds since $b \models Kq$, so $b \not\models \neg Kq$, and so $a \not\models K\neg Kq$. Hence $a \not\models q \vee K\neg Kq$, and $\text{Ref} \cap \text{Trans} \not\models q \vee K\neg Kq$. By the above stated completeness theorem for S_4 we thus obtain $\not\models_{S_4} q \vee K\neg Kq$.

(c) $\vdash_{S5} \neg K K p \rightarrow K \neg K p$ (5 pt)

We give a derivation in *S5*:

1. $Kp \rightarrow KKp$ (positive introspection)
2. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ (tautology)
3. $(Kp \rightarrow KKp) \rightarrow (\neg KKp \rightarrow \neg Kp)$ (subst. instance of 1)
4. $\neg KKp \rightarrow \neg Kp$ (modus ponens, 3, 1)
5. $\neg Kp \rightarrow K\neg Kp$ (negative introspection)
6. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ (tautology)
7. $4 \rightarrow (5 \rightarrow (\neg KKp \rightarrow K\neg Kp))$ (subst. instance of 8)
8. $5 \rightarrow (\neg KKp \rightarrow K\neg Kp)$ (modus ponens, 7, 4)
9. $\neg KKp \rightarrow K\neg Kp$ (modus ponens, 8, 5)

(Alternatively we can show that the formula is valid in all frames (W, R) , where R is an equivalence relation, and use the completeness theorem for system $S5$ to conclude that the formula is derivable in $S5$.)