Exam Advanced Logic

VU University Amsterdam, 3 June 2013, 18:30–21:15

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed. In total you can score 90 points as indicated per question. The final grade is the minimum of 10 and (points/10 + 1 + bonus).

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1. **(a)** Define what it means that a modal formula is globally true in a model. *(3 pt)*

(b) Define what it means that a modal formula is valid in a frame. *(3 pt)*

Consider the frame $\mathfrak{F} = (W, R)$ with W and R given by

$$W = \{a, b, c, d\} \qquad\qquad R = \{(a, b), (b, c), (c, a), (d, a), (d, c)\}$$

and the model $\mathfrak{M} = (\mathfrak{F}, V)$ with valuation V defined by $V(p) = \{a, c\}$.

(c) Give a graphical representation of $\mathfrak{M}$. *(2 pt)*

(d) Prove that $p \to \Box\Box\Box p$ is globally true in $\mathfrak{M}$, but not valid in $\mathfrak{F}$. *(3+5 pt)*

(e) Prove that for any formula φ, the formula $\Box\varphi \leftrightarrow \Box\Box\Box\Box\varphi$ is valid in $\mathfrak{F}$. *(8 pt)*

2. **(a)** Let I be an arbitrary index set, and let $h, i, j \in I$. Prove that the formula $[i][j]p \to [h]p$ characterizes the class of I-frames $\mathfrak{F} = (W, \{R_k \mid k \in I\})$ that satisfy the property $R_h \subseteq R_i \circ R_j$. (Here $R \circ S$ denotes the composition of relations R and S.) *(10 pt)*

(b) Prove that the formula $\Box\Box p \to \Box p$ is not valid in the frame $(\mathbb{Z}, <)$, the integer numbers ordered by the usual less-than relation $<$. *(7 pt)*

(c) Prove that the formula $\Box\Box p \to \Box p$ is valid in the frame $(\mathbb{Q}, <)$, the rational numbers ordered by the usual less-than relation $<$. *(7 pt)*

3. System T is the extension of the minimal modal logic K with the axiom of veridicality (if something is known, it is true). System $S4$ extends T with the axiom of positive introspection; $S5$ extends $S4$ with the axiom of negative introspection. Assume there are $n \geq 2$ agents.

Prove or disprove the following epistemic claims (you may use completeness theorems):

(a) $\vdash_T K_1K_2p \rightarrow K_2\neg K_1\neg p$ *(5 pt)*

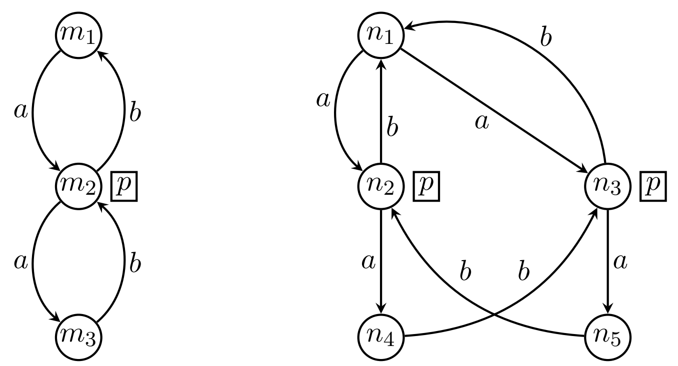
(b) $\vdash_{S4} \neg q \rightarrow K_2\neg K_2q$ *(5 pt)*

(c) $\vdash_{S5} \neg K_2K_2p \rightarrow K_2\neg K_2p$ *(5 pt)*

The system K^+ is the extension of K with the axioms for common knowledge, and the induction axiom $C(p \rightarrow Ep) \rightarrow (Ep \rightarrow Cp)$.

(d) Give a derivation to show $\vdash_{K^+} ECp \rightarrow CCp$. *(5 pt)*

4. Consider the $\{a, b\}$-models $\mathcal{A}$ and $\mathcal{B}$ defined by:



$\mathcal{A}$ $\qquad$ $\mathcal{B}$

(a) Define model $\mathcal{A}$ by means of set notation. *(2 pt)*

(b) Is there a modal formula that distinguishes state n_3 in model $\mathcal{B}$ from state m_2 in model $\mathcal{A}$? Prove your answer. *(8 pt)*

(c) Let $\widehat{\mathcal{B}}$ be the PDL-extension of model $\mathcal{B}$. Compute the transition relation $\widehat{R}_\pi$ corresponding to the PDL-program $\pi = \textsf{if } p \textsf{ then } ba \textsf{ else } ab$. *(8 pt)*

(d) Determine whether the PDL-formula $[\pi]p \rightarrow \langle b\rangle\top$ globally holds in $\widehat{\mathcal{B}}$. Prove your answer. *(4 pt)*