Exam Advanced Logic

VU University Amsterdam, 26 March 2013, 15:15–18:00

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed.
In total you can score 90 points as indicated per question. The final grade is the minimum of 10 and (points/10 + 1 + bonus).

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1. **(a)** Define what it means that a modal formula is valid in a class of frames. (Give your answer in terms of the notion of truth of a formula in a point of a model.) *(4 pt)*

(b) Prove that the formula $\Box\Diamond p \to \Box\Diamond\Box\Diamond p$ is valid in all transitive frames. *(8 pt)*

(c) Can the formula of the previous item also be valid in a non-transitive frame? Prove your answer. *(5 pt)*

(d) Show that $\mathfrak{F} \models \Diamond r \to \Box r$ implies $\mathfrak{F} \models (\Box p \to \Box q) \to \Box(p \to q)$, for all frames $\mathfrak{F}$. (Here p, q, r are proposition variables.) *(8 pt)*

2. For $n = 1, 2, 3, \ldots$, let the 'looping frame' $\mathcal{L}_n = (W_n, R_n)$ be defined by

$$W_n = \{0, \ldots, n-1\}$$
$$R_n = \{\, (k, k') \mid k' = k+1 \text{ if } k+1 < n \text{ and } k' = 0 \text{ otherwise} \,\}$$

(a) Draw the frames $\mathcal{L}_2$ and $\mathcal{L}_4$. *(2 pt)*

(b) Give a modal formula that distinguishes frame $\mathcal{L}_2$ from $\mathcal{L}_4$, that is, a formula φ such that $\mathcal{L}_2 \models \varphi$ and $\mathcal{L}_4 \not\models \varphi$. Prove your answer. *(8 pt)*

For questions **(c)** and **(d)** you have to define a bisimulation, but reporting on the verification of the bisimulation conditions is not required.

(c) Let $\mathcal{M}_2$ be some model based on $\mathcal{L}_2$. Define a model $\mathcal{M}_4$ based on $\mathcal{L}_4$ such that $\mathcal{M}_2, 0 \leftrightarroweq \mathcal{M}_4, 0$. *(6 pt)*

(d) Let $\mathcal{M}_3$ be some model based on $\mathcal{L}_3$. Define an acyclic model $\mathcal{N}$ bisimilar to $\mathcal{M}_3$. *(6 pt)*

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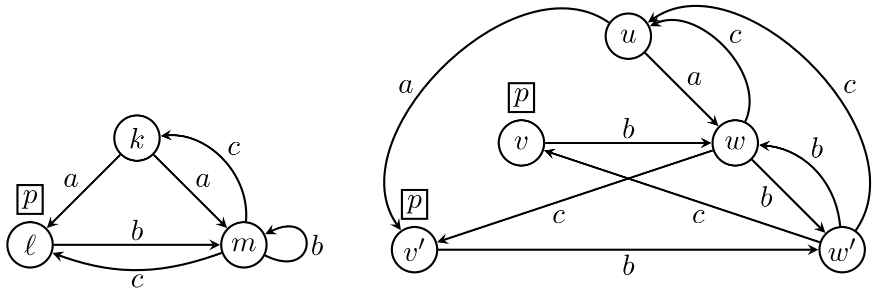
3. Consider the $\{a, b, c\}$-models $\mathcal{A}$ and $\mathcal{B}$ defined by:



$\mathcal{A}$ $\qquad\qquad$ $\mathcal{B}$

(a) Is there a modal formula that distinguishes state k in model $\mathcal{A}$ from state u in model $\mathcal{B}$? Prove your answer. *(8 pt)*

(b) Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\alpha$ corresponding to the PDL-program $\alpha = \textsf{while } \neg p \textsf{ do } a \cup bc$. *(8 pt)*

(c) Determine whether the PDL-formula $[\alpha]p \to \langle b \rangle \top$ globally holds in $\widehat{\mathcal{A}}$. Prove your answer. *(4 pt)*

4. System T is the extension of the minimal modal logic K with the axiom of veridicality (if something is known, it is true). System $S4$ extends T with the axiom of positive introspection; $S5$ extends $S4$ with the axiom of negative introspection. Assume there are $n \geq 2$ agents.

(a) Prove or disprove the following epistemic claims:

(i) $\vdash_T K_1 K_2 p \to K_2 \neg K_1 \neg p$ *(5 pt)*

(ii) $\vdash_{S4} \neg K_1 K_1 p \to K_1 \neg K_1 p$ *(5 pt)*

(iii) $\vdash_{S5} \neg K_2 K_2 p \to K_2 \neg K_2 p$ *(5 pt)*

(b) Show that validity of the axiom $Cp \to ECp$ in an epistemic frame forces that the frame has the property $R_E; R_C \subseteq R_C$. (Recall that $R_E; R_C$ denotes the relational composition of R_E and R_C.) *(8 pt)*