# Exam Advanced Logic

Vrije Universiteit Amsterdam, 4 June 2012, 18:30–21:15

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed.

1. (a) Prove or disprove the validity of $\Box\Diamond\Box p \to \Diamond p$ in reflexive frames.

   (b) Prove or disprove the validity of $p \to \Diamond\Box p$ in reflexive, transitive frames.

   (c) Is the formula $\Box p \to \Box\Box\Box p$ valid in transitive frames? Is it valid only in transitive frames? Prove your answers.

   (d) Show that the frame property $\forall xy\,(Rxy \to x = y)$ is characterized by the modal formula $p \to \Box p$.

2. Consider the extension of the basic modal language with the 'difference' operator D whose semantics is given by:

   $$\mathcal{M}, s \vDash D\varphi \quad \text{if and only if} \quad \text{there exists } t \neq s \text{ such that } \mathcal{M}, t \vDash \varphi.$$

   Let $\mathcal{F}$ en $\mathcal{G}$ be the frames defined by the following pictures:
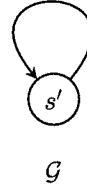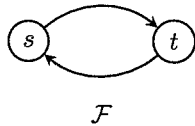
   (a) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{F}$? Explain your answer.

   (b) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{G}$? Explain your answer.

   (c) Show that the formula $\Diamond p \to Dp$ characterizes the class of irreflexive frames.

   (d) Show that D is not definable in the basic modal language.

3. Consider the $\{a, b, c\}$-models $\mathcal{A}$ and $\mathcal{B}$ defined by:

$$\mathcal{A} \qquad\qquad\qquad \mathcal{B}$$
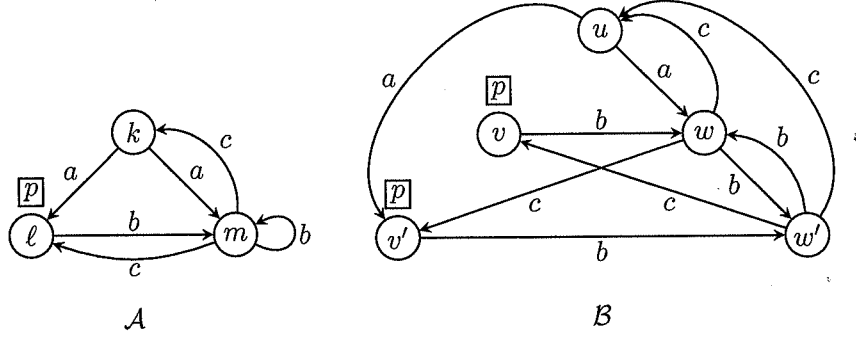
(a) Define a relation $E \subseteq \{k, \ell, m\} \times \{u, v, w, v', w'\}$ such that $E : \mathcal{A}, k \underleftrightarrow{} \mathcal{B}, u$. Show that E satisfies the forward condition (*zig*) of bisimulations.

(b) Is there a formula which is true in state m but false in state w? Explain.

(c) Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\alpha$ corresponding to the PDL-program α:

$$\alpha = \text{if } p \text{ then } (bc)^* \text{ else } (a \cup c)^*$$

(d) Determine for each of the PDL-formulae $\langle \alpha \rangle p$ and $[\alpha]p$ whether it holds in $\widehat{\mathcal{A}}$.

4. The Hilbert system T is the extension of the system K with the axiom of veridicality $A1$ (if something is known, it is true). System $S4$ is the extension of T with $A2$, the axiom of positive introspection. System $S5$ is the extension of $S4$ with $A3$, the axiom of negative introspection.

(a) Formulate the completeness theorem for the systems T and $S5$.

(b) Give a derivation or construct a countermodel:

    (i) $\vdash_T (K(p \to q) \wedge p) \to q$

    (ii) $\vdash_{S4} \neg KKp \to K\neg Kp$

    (iii) $\vdash_{S5} \neg KKp \to K\neg Kp$