Exam Advanced Logic

Vrije Universiteit Amsterdam, 28 March 2012, 08:45–11:30

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed.

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1. (a) Prove or disprove the validity of $\Box\Diamond\Box p \to \Diamond p$ in reflexive frames.
   (b) Show that $\mathfrak{F} \models \Diamond r \to \Box r$ implies $\mathfrak{F} \models (\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$.
   (c) Is the formula $\Box p \to \Box\Box\Box p$ valid in transitive frames? Is it valid only in transitive frames? Prove your answers.
   (d) Show that validity of the formula $\Diamond\Box p \to p$ in a frame $\langle W, R\rangle$ implies that R is symmetric.

2. Consider the frame $\mathfrak{F} = \langle H, R\rangle$ with $H = \{1, 2, \ldots, 24\}$ and $R = \{\langle 1, 8\rangle, \langle 2, 9\rangle, \ldots, \langle 17, 24\rangle, \langle 18, 1\rangle, \ldots, \langle 23, 6\rangle, \langle 24, 7\rangle\}$, so that the future of h is precisely 7 hours later.

   We extend the basic temporal language with an operator pm (for *'post meridiem'*) with the following semantics:

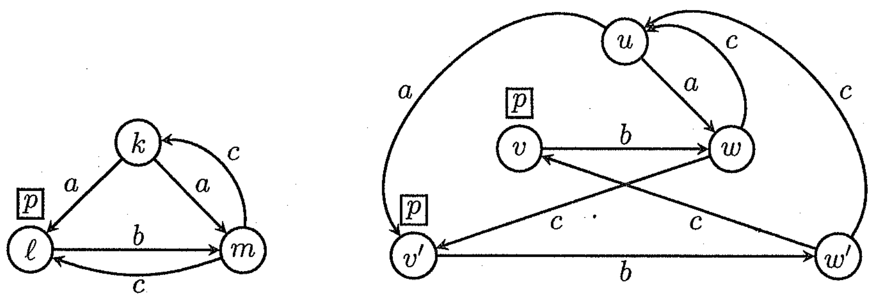
$$\langle \mathfrak{F}, V\rangle, h \models \mathsf{pm}\,\varphi \iff h \in \{13, 14, \ldots, 23, 24\} \text{ and } \langle \mathfrak{F}, V\rangle, h \models \varphi$$

   (a) Show that the formulas $\langle F\rangle p \to \langle F\rangle\langle F\rangle p$ and $\neg(\langle F\rangle p \to \langle F\rangle\langle F\rangle p)$ are not valid in $\mathfrak{F}$.
   (b) Show that for all models $\mathfrak{M}$ on the frame $\mathfrak{F}$ we have the following equivalence:

$$\mathfrak{M}, h \models \langle F\rangle\mathsf{pm}\,\langle P\rangle\varphi \iff h \in \{6, 7, \ldots, 17\} \text{ and } \mathfrak{M}, h \models \varphi$$

(see next page)

3. Consider the following models $\mathcal{A}$ and $\mathcal{B}$:



$\mathcal{A}$ $\mathcal{B}$

(a) Define a relation $E \subseteq \{k, \ell, m\} \times \{u, v, w, v', w'\}$ such that $E : \mathcal{A}, k \leftrightarroweq \mathcal{B}, u$. Only show that E satisfies the backwards condition (*zag*) of bisimulations.

(b) Is there a formula which is true in state k but false in state u? Explain.

(c) Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\alpha$ corresponding to the PDL-program α:

$$\alpha = \textsf{if } p \textsf{ then } (bc)^* \textsf{ else } (a \cup c)^*$$

(d) Determine for each of the PDL-formulae $\langle\alpha\rangle p$ and $[\alpha]p$ whether it holds in $\widehat{\mathcal{A}}$.

4. The proof system $S4$ is the extension of the base system K with the axioms of veridicality (A1) and positive introspection (A2).

(a) Give a derivation for $\vdash_{S4} K\neg KKp \to K\neg Kp$.

(b) Show that the axiom of negative introspection (A3) is not derivable in $S4$.

(c) A relation R is *euclidian* if $Rxy \wedge Rxz \to Ryz$. Show that a reflexive and euclidian relation is symmetric.

(d) Consider the proof system S obtained by extending the basic proof system K with the axioms of veridicality (A1) and of negative introspection (A3). Show that the formula $p \to K\neg K\neg p$ is derivable in S.