Exam Advanced Logic

Vrije Universiteit Amsterdam, 30 May 2011, 18:30–21:15

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed.

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1. (a) Prove or disprove the validity of $\Box\Diamond\Box p \to \Diamond p$ in reflexive frames.

   (b) Prove or disprove the validity of $p \to \Diamond\Box p$ in reflexive, transitive frames.

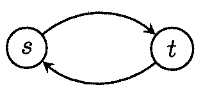
   (c) Is the formula $\Box p \to \Box\Box\Box p$ valid in transitive frames? Is it valid only in transitive frames? Prove your answers.

   (d) Show that the frame property $\forall xy\,(Rxy \to x = y)$ is characterized by the modal formula $p \to \Box p$.

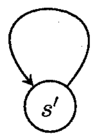
2. Consider the extension of the basic modal language with an operator D whose semantics is given by:

$$\mathcal{M}, s \models D\varphi \iff \text{there exists } t \neq s \text{ such that } \mathcal{M}, t \models \varphi\,.$$

Let $\mathcal{F}$ en $\mathcal{G}$ be the frames defined by the following pictures:



$\mathcal{F}$



$\mathcal{G}$

   (a) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{F}$? Explain your answer.

   (b) Is the formula $\Diamond p \to Dp$ valid in $\mathcal{G}$? Explain your answer.

   (c) Show that the formula $\Diamond p \to Dp$ characterizes the class of irreflexive frames.

   (d) Show that D is not definable in the basic modal language.

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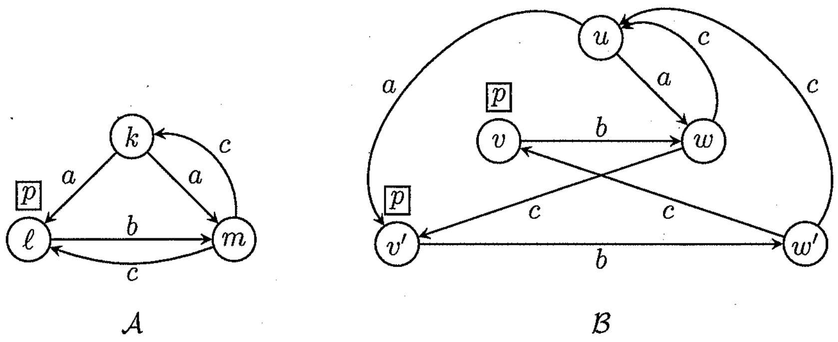
3. Consider the $\{a, b, c\}$-models $\mathcal{A}$ and $\mathcal{B}$ defined by:



(a) Define a relation $E \subseteq \{k, \ell, m\} \times \{u, v, w, v', w'\}$ such that $E : \mathcal{A}, k \leftrightarrows \mathcal{B}, u$. Only show that E satisfies the backwards condition (*zag*) of bisimulations.

(b) Is there a formula which is true in state k but false in state u? Explain.

(c) Let $\widehat{\mathcal{A}}$ be the PDL-extension of model $\mathcal{A}$. Compute the transition relation $\widehat{R}_\alpha$ corresponding to the PDL-program α:

$$\alpha = \mathsf{if}\ p\ \mathsf{then}\ (bc)^*\ \mathsf{else}\ (a \cup c)^*$$

(d) Determine for each of the PDL-formulae $\langle\alpha\rangle p$ and $[\alpha]p$ whether it holds in $\widehat{\mathcal{A}}$.

4. The Hilbert system T is the extension of the system K with the truth axiom $A1$. System $S4$ is the extension of T with $A2$, the axiom of positive introspection.

(a) Formulate the completeness theorem for the system $S4$.

(b) Show that $A3$, the axiom of negative introspection, is not deducible in $S4$.

(c) Prove that a reflexive and euclidian relation is an equivalence relation.

(d) Give a derivation or construct a countermodel:

(i) $\vdash_{K^2} K_1 K_2 p \to K_2 K_1 p$

(ii) $\vdash_T (K(p \to q) \wedge p) \to q$