Exam Advanced Logic

Vrije Universiteit Amsterdam, 25 March 2011, 08:45–11:30

This exam consists of four questions. Use of textbook, definition sheets, etc. is not allowed.

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1. **(a)** Prove or disprove the validity of $\Box p \to \Diamond\Box\Diamond p$ in reflexive frames.

(b) Prove or disprove the validity of $\Box\Diamond p \to p$ in reflexive, transitive frames.

(c) Give a poly-modal formula which characterizes the property $R_1 \circ R_2 \subseteq R_2 \circ R_1$, and prove that it does so.

(d) Is it possible that a frame property is characterized by non-equivalent formulas? Put differently, is there a class of frames which is modally defined by both φ and ψ, but with $\varphi \leftrightarrow \psi$ not universally valid? Explain your answer.

2. Let $A = \{a_0, a_1, a_2 \ldots\}$ be a set of atomic programs, and consider the PDL-frame $\mathcal{F} = (\mathbb{N}, \{R_\alpha \mid \alpha \in PROG(A)\})$ with the accessibility relations for the atomic programs given by $R_{a_i} = \{\, (k, k+i) \mid k \geq 0 \,\}$.

(a) Prove that $\mathcal{F} \vDash \langle a_i \rangle [a_j] p \to [a_j] \langle a_i \rangle p$.

Consider the model $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ with $\mathcal{V}(p) = \mathbb{N} \setminus \{2011\}$, and the PDL-program $\beta = (a_3 \cup a_2)^*; a_0$.

(b) Determine the accessibility relation for β.

(c) Is the PDL-formula $[\beta]p$ globally true in $\mathcal{M}$? Explain your answer.

(d) Is $[\beta]p \to [\beta\beta]p$ valid in $\mathcal{F}$? Prove your answer.

(see next page)

3. Let $n \in \mathbb{N}$ and consider the 'looping frame' $\mathcal{L}_n = (W_n, R_n)$ with

$$W_n = \{0, \ldots, n-1\}$$
$$R_n = \{\, (k, k') \mid k' = k+1 \text{ if } k+1 < n \text{ and } k' = 0 \text{ otherwise} \,\}$$

(a) Give a distinguishing formula for every two distinct looping frames, that is, for all $n, m \in \mathbb{N}$ with $n \neq m$ a formula φ valid in $\mathcal{L}_n$ but not in $\mathcal{L}_m$.

(b) Show that for every model $\mathcal{M}_n$ based on $\mathcal{L}_n$ there exists a bisimilar model $\mathcal{M}_{2n}$ based on $\mathcal{L}_{2n}$.

(c) For any model $\mathcal{N}$, does there exist a model $\mathcal{N}'$ bisimilar to $\mathcal{N}$ such that the accessibility relation of $\mathcal{N}'$ is acyclic? Explain your answer.

4. Prove or disprove the following epistemic claims. Here K is the basic epistemic logic, and $S5$ the extension of K with the axioms of veridicality, positive and negative introspection. The system K^+ is the extension of K with the axioms for common knowledge, and the induction axiom $C(p \to Ep) \to (Ep \to Cp)$. Assume that there are $n \geq 2$ agents.

(a) $\vdash_{S5} K_1 K_2 p \wedge K_1(p \to q) \to K_1 \neg K_2 q$

(b) $\vdash_{K} K_1 K_2 p \to K_1 K_1 p$

(c) $\vdash_{K} (K_1 p \vee K_1 q) \wedge K_1(p \to q) \to K_1 q$

(d) $\vdash_{K^+} ECp \to CCp$