Examination

Advanced Linear Programming
Spring Semester 2023

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Information

General Information

- The examination lasts 180 minutes.
- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Part 1 has 5 questions and part 2 has 3 questions.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.
- Give your answers to the two parts on separate sheets!!!

GOOD LUCK

Part 1

Exercise 1 1 point

Formulate Farkas' Lemma.

Solution: **Theorem.** Given $m \times n$ matrix A and $b \in \mathbb{R}^m$, exactly one of the following two alternatives holds:
a) $\exists\, x \geq 0 :\ Ax = b$;
b) $\exists\, y \in \mathbb{R}^m :\ y^T A \geq 0 \wedge y^T b < 0$.

Exercise 2 1 point

Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. Prove that exactly one of the following holds:

(a) there exist $x \in \mathbb{R}^n$ such that $Ax = b$;

(b) there exist $y \in \mathbb{R}^m$ such that $y^T A = 0$ and $y^T b < 0$.

Solution: To turn (a) in the form used in Farkas Lemma as in Exercise 1, we introduce for each variable x_i two non-negative variables: $x_i^+ \geq 0$ and $x_i^- \geq 0$ and replace each variable x_i by $x_i^+ - x_i^-$, $i = 1, \ldots, n$. Using the notation x^+ and x^- for the vectors in $\mathbb{R}^n$ with coordinates x_i^+ and x_i^- resp., this yields the equivalent expression for (a)

(a) there exist $x^+ \in \mathbb{R}^n$ and $x^- \in \mathbb{R}^n$, such that $Ax^+ - Ax^- = b$ and $x^+ \geq 0$ and $x^- \geq 0$.
According to Farkas Lemma the corresponding statement for (b) should then be

(b) there exists $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T(-A) \geq 0$ and $y^T b < 0$,

which is equivalent to

(b) there exists $y \in \mathbb{R}^m$ such that $y^T A = 0$ and $y \geq 0$ and $y^T b < 0$.

QED

Also correct proofs from scratch have been awarded full points.

Exercise 3 0.5+0.5+0.5 points

Given the LP $\min\{c^T x \mid Ax = b, x \geq 0\}$. From basic feasible solution x we make a step θd for some positive value θ and a vector d such that $d_j = 1$, for non-basic variable x_j, $d_j = 0$ for all other non-basic variables and $Ad = 0$.

State for each of the following statements if they are TRUE or FALSE. A correct answer gives $+0.5$ points. An incorrect answer gives -0.5 points. No answer gives 0 points.

(a) If $d \geq 0$ then the feasible region of the LP-problem is unbounded.

(b) If $c^T d < 0$ then the optimal solution value is $-\infty$.

(c) If $d \ngeq 0$, then x_j will be made basic variable. This implies that in the very next iteration of the simplex method it cannot leave the basis.

Solution:

(a) True

(b) False

(c) False

Exercise 4 1+0.5 points

Consider the max-flow problem on the network in Figure 1, after having done already several Ford-Fulkerson flow augmentations. The first number next to an arc gives the capacity of that arc. The second number next to an arc gives the current flow on that arc. If a second number next to an arc is missing, it means that there is currently no flow on that arc.

a) Find the maximum flow starting from the current flow and applying flow augmentation. If you need to draw graphs to find your solution, you can make use of the empty graphs on the next page.
NB: Not starting with the given flow will give no points at all!!!

b) Prove that your answer is correct.

Solution:

a) Under the current flow there is no flow augmenting path using only forward arcs. Thus, construct the residual graph, which has a.o. the arcs $(2,4)$ and $(4,1)$, allowing for the augmenting path $s,2,4,1,3,6,t$ with bottleneck capacity 1 on arc $(1,3)$. Fill in the right flow numbers and conclude that in the new residual graph there is no further augmenting path. We have reached a flow of 8.

b) Given the residual graph constructed last the vertices that can still be reached from s are $2,4,1,5,8$. This yields the cut consisting of the arcs $(1,3),(4,3),(4,6)$ and $(8,t)$ with a total capacity of 8.

Since it is obvious that the total flow cannot be more than the capacity of any cut, this gives us a certificate of optimality of the answer given.

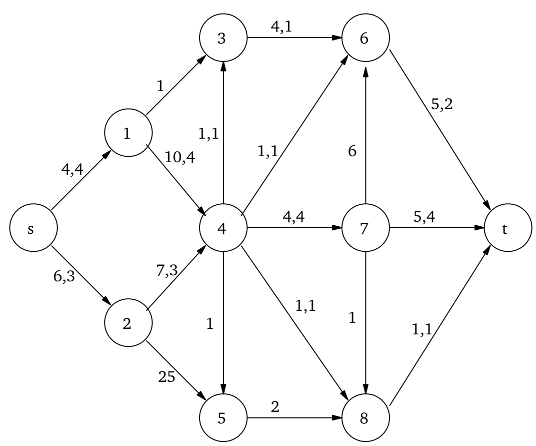

Figure 1: Network with flow

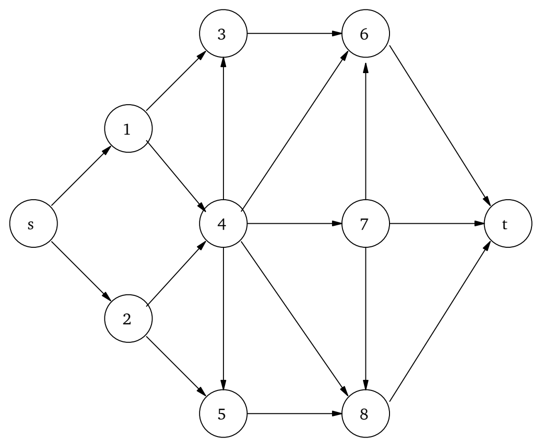

Figure 2: Empty Network 1

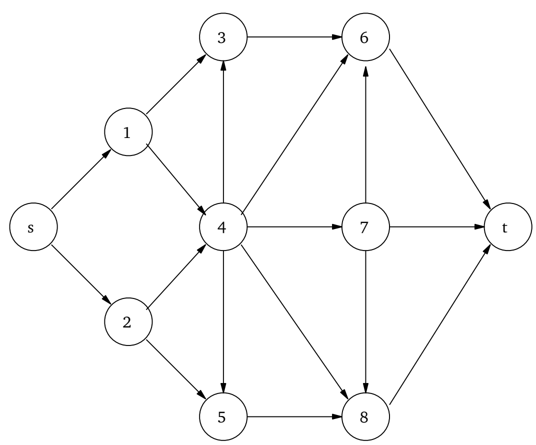

Figure 3: Empty Network 2

Part 2

Exercise 5 0.5+1 points

Let $G = (V, E)$ be an undirected graph. A set $S \subseteq V$ is called *stable* if, for all distinct $u, v \in S$, we have $\{u, v\} \notin E$. That is, no pair of nodes in S is connected by an edge. Consider the following model:

$$\max \sum_{v \in V} x_v \tag{1a}$$

$$x_u + x_v \leq 1, \qquad \{u, v\} \in E, \tag{1b}$$

$$x_v \leq 1, \qquad v \in V \tag{1c}$$

$$x_v \geq 0, \qquad v \in V, \tag{1d}$$

$$x_v \in \mathbb{Z}, \qquad v \in V. \tag{1e}$$

1. Show that (1) is an integer programming model for finding a stable set of maximum cardinality. To this end, show that
   (a) for every stable set S in G, there exists a feasible solution x of (1) whose objective is $|S|$.
   (b) for every feasible solution x of (1), there exists a stable set S in G with $|S| = \sum_{v \in V} x_v$.
2. The graph $G = (V, E)$ is called *bipartite* if there exist disjoint sets $A, B \subseteq V$ with $V = A \cup B$ such that, for every $a \in A$ and $b \in B$, we have $\{a, b\} \notin E$. Let

   $$P(G) = \{x \in \mathbb{R}^V : x \text{ satisfies (1b), (1c), (1d)}\}.$$

   Prove that every vertex of $P(G)$ is integral if G is bipartite.

Solution:

1. (a) Let S be a stable set in G. Define $x \in \{0, 1\}^V$ by setting $x_v = 1$ if $v \in S$ and $x_v = 0$ otherwise. Obviously, $\sum_{v \in V} x_v = |S|$, so it suffices to show that x is feasible for (1). Obviously, it satisfies (1c), (1d), and (1e). Since S is a stable set, for each edge $\{u, v\} \in E$ we have $|\{u, v\} \cap S| \leq 1$. Consequently, (1b) holds, i.e., x is feasible for (1).
   (b) Let x be feasible for (1). Due to (1c), (1d), and (1e), we have $x \in \{0, 1\}^V$. We claim that $S = \{v \in V : x_v = 1\}$ is a stable set of size $\sum_{v \in V} x_v$. The latter follows immediately from the definition of S, so it suffices to show that S is stable. Due to (1b), there cannot be an edge $\{u, v\}$ such that both endpoints are contained in S. Hence, S is stable.

2. By a theorem from the lecture, it is sufficient to show that the constraint matrix A of (1) without non-negativity constraints is totally unimodular as the right-hand side vector is integral. Note that A is of type

$$\begin{pmatrix} B \\ I \end{pmatrix},$$

where I is an identity matrix. By another lemma from the lecture, A is totally unimodular if B is totally unimodular. Thus, it suffices to show that B is totally unimodular. As total unimodularity is preserved under transposition of B, we show that $B^\top$ is totally unimodular.

To prove that $C = B^\top$ is totally unimodular, we make use of a theorem from the lecture that says that C is totally unimodular if

- every entry of C is contained in $\{0, \pm 1\}$,
- each column of C (i.e., each row of B) has at most two non-zero entries, and
- there exists a partition (M_1, M_2) of the rows of C (the columns of B) such that for each column j of C (each row j of B), we have

$$\sum_{i \in M_1} C_{ij} - \sum_{i \in M_2} C_{ij} = 0.$$

Obviously, the first two properties hold. For the last property, we exploit that G is bipartite and assign M_1 to be one side of the bipartition and M_2 to be the other. Since each column of C (row of B) has exactly two non-zero entries and columns of C (rows of B) correspond to edges of G, one 1-entry belongs to M_1 and the other to M_2. Hence, also the last conditions holds. This shows that C (and thus B) is totally unimodular, which concludes the proof.

Exercise 6 1+1 points

Consider the knapsack polytope

$$P = \operatorname{conv}\{x \in \{0,1\}^5 : 3x_1 + 3x_2 + 4x_3 + 5x_4 + 6x_5 \leq 14\}.$$

(a) Consider the minimal cover $C = \{1, 2, 3, 4\}$. Derive a lifted cover inequality for C.

(b) Let $a \in \mathbb{Z}_+^n$ and let $b \in \mathbb{Z}_+$. Let $X = \{x \in \{0,1\}^n : \sum_{i=1}^n a_i x_i \leq b\}$ and let C be a cover for the knapsack inequality $\sum_{i=1}^n a_i x_i \leq b$. Moreover, let $\bar{a} = \max\{a_i : i \in C\}$ and let $D = \{i \in \{1, \ldots, n\} : a_i > \bar{a}\}$. Prove that

$$\sum_{i \in C} x_i + \sum_{i \in D} x_i \leq |C| - 1$$

is valid for X.

Solution:

1. Since only item 5 is not contained in the cover, it is sufficient to compute the lifting coefficient of x_5 in the cover inequality

$$x_1 + x_2 + x_3 + x_4 \leq 3.$$

To this end, we need to solve the knapsack problem

$$\max\{x_1 + x_2 + x_3 + x_4 : 3x_1 + 3x_2 + 4x_3 + 5x_4 \leq 14 - 6x_5 = 8,\ x_5 = 1\}$$

whose optimal value is 2. The lifting coefficient is then computed as $|C| - 2 = 1$. The lifted inequality is

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 3.$$

2. Suppose the inequality was not valid. Then, there would exist $x \in X$ with

$$\sum_{i \in C} x_i + \sum_{i \in D} x_i \geq |C|.$$

W.l.o.g. we can assume $\sum_{i \in C} x_i + \sum_{i \in D} x_i = |C|$, because turning some 1-entries in x to 0 cannot violate the knapsack inequality defining X. Since the cover inequality $\sum_{i \in C} x_i \leq |C| - 1$ is valid for X, not all elements in C can be 1-entries of x. This means that there exist $j \in D$ and $j' \in C$ such that $x_j = 1$ and $x_{j'} = 0$.

We define an alternative solution $\bar{x}$ via

$$\bar{x}_i = \begin{cases} 1, & \text{if } i = j', \\ 0, & \text{if } i = j, \\ x_i, & \text{otherwise.} \end{cases}$$

Note that $\bar{x} \notin X$ since $\bar{x}_i = 1$ for every $i \in C$, i.e., it violates the associated cover inequality. Since $a_j \geq \bar{a} \geq a_{j'}$, we thus find

$$b < \sum_{i=1}^{n} a_i \bar{x}_i \leq \sum_{i=1}^{n} a_i x_i.$$

Thus, also $x \notin X$, contradicting our initial assumption. For this reason,

$$\sum_{i \in C} x_i + \sum_{i \in D} x_i \leq |C| - 1$$

needs to be valid for X.

Exercise 7 0.5+1 points

Consider the following example of a cutting stock problem. We are given large paper rolls of width 10 from which we need to cut 7 paper rolls of width 3, 10 paper rolls of width 4, and 7 paper rolls of width 5. Our goal is to minimize the number of large paper rolls that we need to produce the smaller paper rolls.

Let $J = \{3, 4, 5\}$. Recall that we can model this problem by making use of patterns. A *pattern* is a vector $p \in \mathbb{Z}_+^J$ such that $3p_3 + 4p_4 + 5p_5 \leq 10$. That is, a pattern p encodes a possible way to cut a large paper roll into smaller paper rolls, where entry p_j, $j \in J$, describes how many small paper rolls of width j are cut from a large paper roll. Let P be the set containing all patterns. Then, the cutting stock problem can be modeled as

$$\min \sum_{p \in P} z_p$$
$$\sum_{p \in P} p_3 z_p = 7,$$
$$\sum_{p \in P} p_4 z_p = 10,$$
$$\sum_{p \in P} p_5 z_p = 7,$$
$$z \in \mathbb{Z}_+^P.$$

Since this problem contains many variables, its *LP relaxation* is usually solved by means of column generation. A basis of the *LP relaxation* consists of the patterns

$$a = (3, 0, 0), \qquad b = (0, 2, 0), \qquad c = (0, 0, 2).$$

Your task is to execute one step of the column generation procedure. To this end,

1. Determine the basic feasible solution corresponding to the basis a, b, c.
2. Either prove that the basic feasible solution is optimal or find a pattern with negative reduced cost. Justify your argumentation, e.g., by explaining how the reduced costs can be computed for this particular application.

Solution:

1. The constraint system (without non-negativity constraints) restricted to the basis is

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} z_a \\ z_b \\ z_c \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ 7 \end{pmatrix},$$

whose solution is given by $(z_a, z_b, z_c)^\top = (\frac{7}{3}, 5, \frac{7}{2})$.

2. To prove of refute optimality, we need to solve the pricing problem. The general formula for the reduced cost of variable p is

$$c_p - c_B A_B^{-1} A_{\cdot p} = 1 - (\frac{1}{3}, \frac{1}{2}, \frac{1}{2}) A_{\cdot p}.$$

Since column $A_{\cdot p}$ corresponds to the pattern p, we need to check whether there exists a pattern p such that $\frac{1}{3}p_3 + \frac{1}{2}p_4 + \frac{1}{2}p_5 > 1$. To this end, we are looking for a pattern maximizing this expression, which can be found by solving

$$\max\{\frac{1}{3}p_3 + \frac{1}{2}p_4 + \frac{1}{2}p_5 : 3p_3 + 4p_4 + 5p_5 \leq 10, p \in \mathbb{Z}_+^{\{3,4,5\}}\}.$$

An optimal solution of this problem is given by $p = (2, 1, 0)$ with objective value $\frac{7}{6} > 1$. Hence, the current basis is not optimal as the variable for pattern p has negative reduced cost.