Examination

Advanced Linear Programming
Spring Semester 2022

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Information

General Information

- The examination lasts 180 minutes.
- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- Part 1 has 5 questions and part 2 has 3 questions.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down and provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.
- Give your answers to the two parts on separate sheets!!!

GOOD LUCK

Part 1

Exercise 1 1 point

Formulate Farkas' Lemma.

Solution: **Theorem.** Given $m \times n$ matrix A and $b \in \mathbb{R}^m$, exactly one of the following two alternatives holds:
a) $\exists\, x \geq 0 : \; Ax = b$;
b) $\exists\, y \in \mathbb{R}^m : \; y^T A \geq 0 \wedge y^T b < 0$.

Exercise 2 1 point

Let A be an $m \times n$ matrix, let C be an $m \times k$ matrix and let $b \in \mathbb{R}^m$. Prove that exactly one of the following holds:

(a) there exist $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$ such that $Ax + Cu = b$ and $x \geq 0$;

(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C = 0$ and $y^T b < 0$.

Solution: PROOF. To turn (a) in the form used in Farkas Lemma as in Exercise 1, we introduce $u^1 \geq 0$ and $u^2 \geq 0$ with $u^1, u^2 \in \mathbb{R}^k$ allowing us to express any $u \in \mathbb{R}^k$ as $u = u^1 - u^2$. This yields the equivalent expression for (a)
(a) there exist $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^k$ such that $Ax + Cu^1 - Cu^2 = b$ and $x, u^1, u^2, s \geq 0$;
According to Farkas Lemma the corresponding statement for (b) should then be
(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C \geq 0$, $y^T(-C) \geq 0$ and $y^T b < 0$,
which is equivalent to
(b) there exist $y \in \mathbb{R}^m$ such that $y^T A \geq 0$, $y^T C = 0$ and $y^T b < 0$.
QED

Also correct proofs from scratch have been awarded full points.

Exercise 3 0.75 points

Consider the simplex method applied to a standard form LP-problem

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

with $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A a $m \times n$ matrix. Assume that the m rows of the matrix A are linearly independent. For each of the statements that follow, indicate if it is TRUE or FALSE. Argue your answers briefly (you don't need to prove them). A correct answer gives 0.25 pt., an incorrect answer gives -0.25 pt. No answer gives 0 pt. The total score will always be non-negative.

(i) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.

(ii) A variable that has just left the basis cannot reenter in the very next iteration.

(iii) A variable that has just entered the basis cannot leave in the very next iteration.

Solution:

(i) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
Answer. FALSE: in each pivot step a direction is chosen in which there is strict improvement.

(ii) A variable that has just left the basis cannot reenter in the very next iteration.
Answer. TRUE: in the pivot step the leaving variable gets a positive reduced objective coefficient.

(iii) A variable that has just entered the basis cannot leave in the very next iteration.
Answer. FALSE: A simple drawing (that I do not make here) can show this.

Exercise 4 0.75+0.25 points

Hint: part (b) of this exercise is easier than part (a) and can be made without having part (a) solved correctly.

Given is the following theorem.

Theorem 0.1. *Let $a_1, \ldots, a_m$ be some vectors in $\mathbb{R}^n$, with $m > n+1$. Suppose that the system of inequalities $a_i^T x \geq b_i$, $i = 1, \ldots, m$, does not have any solutions. Then we can choose $n+1$ of these inequalities, so that the resulting system of inequalities has no solutions.*

(a) Use this theorem to prove Helly's Theorem.

Theorem 0.2. Helly's Theorem. *Let $\mathcal{F}$ be a finite family of polyhedra in $\mathbb{R}^n$ such that every $n+1$ polyhedra in $\mathcal{F}$ have a point in common. Then all polyhedra in $\mathcal{F}$ have a point in common.*

(b) For $n = 2$, Helly's Theorem asserts that the polyhedra $P_1, P_2, \ldots, P_K$, $(K \geq 3)$ in the plane have a point in common if and only if every three of them have a point in common. Is the result still true with "three" replaced by "two"?

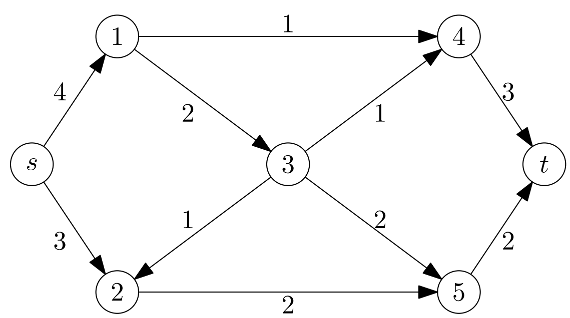
Solution:

(a) Proof by contradiction. Suppose that not all polyhedra in $\mathcal{F}$ have a point in common. Think of the inequalities of all polyhedra together as one big system of inequalities. So there is no point that satisfies all of these inequalities. Then Theorem 0.1 says that there must be a set of $n+1$ of them that is not satisfied by any point. Take such a subset of $n+1$ inequalities. They belong to at most $n+1$ polyhedra of $\mathcal{F}$. Hence the inequalities of these polyhedra cannot all be simultaneously satisfied. But according to the statement of Helly's Theorem any set of $n+1$ polyhedra, hence any set of at most $n+1$ polyhedra of $\mathcal{F}$ has a point in common. Contradiction.

(b) The answer is NO. It is sufficient to draw an example here, or otherwise define 3 sets that pairwise intersect but do not have a point in the intersection of all three sets.

Exercise 5 0.5+0.25+0.5 points

(a) Consider the network in the figure. The number next to an arrow gives the capacity of the arrow. Using Ford-Fulkerson we will determine the maximum flow in this network from node s to node t.



In the first two iterations of Ford-Fulkerson we have found the following flow augmenting paths:

- $s, 1, 4, t$ with bottleneck capacity 1 because of arrow $(1, 4)$;
- $s, 1, 3, 5, t$ with bottleneck capacity 2 because of arrows $(1, 3), (3, 5)$ and $(5, t)$.

So, we have now a flow of size 3 from s to t.

Start from this solution with further iterations of Ford-Fulkerson to find the maximum flow. Also derive the corresponding minimum cut.

(b) (Duality and the max-flow min-cut theorem.) Consider the maximum flow problem, written as the linear program

$$\begin{array}{rl} \max & \sum_{(s,i)\in\mathcal{A}} f_{si}; \\ \text{s.t.} & \sum_{(i,j)\in\mathcal{A}} f_{ij} - \sum_{(j,i)\in\mathcal{A}} f_{ji} = 0, \ \forall i \in \mathcal{N} \setminus \{s,t\}; \\ & 0 \le f_{ij} \le u_{ij}, \ \forall (i,j) \in \mathcal{A}. \end{array}$$

Let p_i be a price variable associated with the flow conservation constraint at node i. Let q_{ij} be a price variable associated with the capacity constraint at arc (i,j). Write down a minimization problem, with variables p_i and q_{ij}, whose dual is the maximum flow problem.

(c) Show that the optimal value in the maximization problem is equal to the minimum cut capacity.

Solution:

(a) In the residual graph there is still the path $s, 2, 5, 3, 4, t$ with capacity 1. This will diminish the flow on arc $(3, 5)$ by 1 and adds 1 to the flow on all other arcs on the path, giving a flow of 4. At this point the residual does not have any s-t path. Thus we have reached the optimum flow. The nodes still reachable from s are s, 2, 5, 3 and 1. They form the cut and indeed this cut has total capacity on the cut-arcs $(1, 4)$, $(3, 4)$ and $(5, t)$ of 4.

(b)

$$
\begin{array}{ll}
\min & \sum_{(i,j)\in\mathcal{A}} u_{ij} q_{ij}; \\
\text{s.t.} & p_i - p_j + q_{ij} \geq 0,\ \forall (i,j) \in \mathcal{A} \setminus \{(s,i),(j,t) \in \mathcal{A}\}; \\
& -p_i + q_{si} \geq 1,\ \forall (s,i) \in \mathcal{A}; \\
& p_j + q_{jt} \geq 0,\ \forall (j,t) \in \mathcal{A}; \\
& g_{ij} \geq 0,\ \forall (i,j) \in \mathcal{A}.
\end{array}
$$

(c) This is literally in the lecture notes.

Part 2

Exercise 6 0.25+0.25+0.6+0.4 points

Consider the integer program

$$\begin{aligned} \max\ & -3x + 2y \\ & -4x + 2y \leq 3 \\ & 3x + 5y \leq 27 \\ & x \leq 4 \\ & x, y \geq 0 \ \text{ and integral.} \end{aligned}$$

(a) Draw the feasible region of the LP relaxation and mark all solutions of the integer program. If the feasible region is unbounded, draw the the part containing all vertices and indicate how the region expands towards infinity. Specify which part of your drawing corresponds to the LP relaxation and which to the integer program.

(b) Using your drawing from a, find an optimal solution of the LP relaxation.

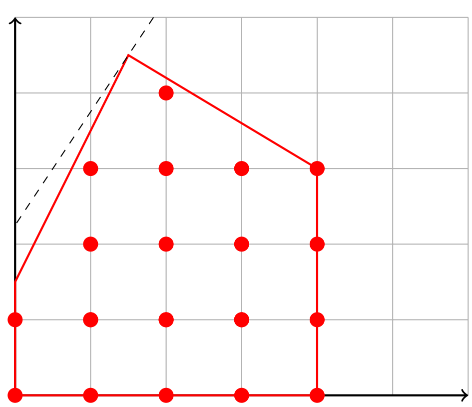
(c) Solve the integer program using branch-and-bound.
Hint: You are allowed to solve the LP relaxations graphically.

(d) The point $(x^\star, y^\star) = (0, \frac{3}{2})$ is an extreme point of the initial LP relaxation of the above integer program. Derive a Gomory cut for $(x^\star, y^\star)$.

Solution:

(a) The picture shows the feasible region of the LP (red lines) and the IP (red dots). (0.15 + 0.1 points)

(If students forget non-negativity constraints or a single other constraint, -0.05 points)



(b) An optimal solution of the LP is given by $(\frac{3}{2}, \frac{9}{2})$.

(c) Since the optimal solution of the LP is not integer, we branch on a fractional variable, e.g., x. This gives us the two subproblems with additional constraints $x \leq 1$ (P1) or $x \geq 2$ (P2).

First, we solve P2, which has $(2, \frac{21}{5})$ as optimal solution. It is again not integral, so we branch on y, which generates the problems with additional constraints $y \leq 4$ (P2a) and $y \geq 5$ (P2b).

The optimal solution of P2a is $(2, 4)$ with objective value 2; P2b is infeasible.

Second, we solve P1, which has $(1, \frac{7}{2})$ as optimal solution. It is again not integral, so we branch on y, which generates the subproblems with additional constraint $y \leq 3$ (P1a) or $y \geq 4$ (P1b).

P1b is infeasible and P1a has optimal solution $(\frac{3}{4}, 3)$ with objective value $\frac{15}{4}$. Since it is still fractional, we continue branching on x, which generates the subproblem with additional constraint $x \leq 0$ (P1a1) and $x \geq 1$ (P1a2).

The optimal solution of (P1a1) is $(0, \frac{3}{2})$ with objective value 3, whereas (P1a2) has optimal solution $(1, 3)$ with objective value 3. Hence, we have found an integral solution with value 3. As all open subproblems have an LP relaxation value of 3, we cannot find a better solution. For this reason, $(1, 3)$ is an optimal solution.

(-0.1 points per mistake) (If students use equations instead of inequalities for branching decisions, 0.1. If they correctly classify infeasible solutions, 0.05)

(d) Note that the first inequality as well as $x \geq 0$ are satisfied with equality by this solution. To derive a Gomory cut for a problem in standard form, we consider

$$-4x + 2y + s = 3$$

where s is a slack variable to turn the initial formulation into standard form. (0.1 points)

A basis for $(0, \frac{3}{2})$ is given by $B = \{y\}$, whereas the non-basis is given by $N = \{x, s\}$. (0.1 points)

The inverse of the basis matrix is then $A_B^{-1} = \frac{1}{2}$. (0.1 points) Hence,

$$\bar{b} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

and the Gomory cut is given by

$$y \leq \lfloor \tfrac{3}{2} \rfloor - (\lfloor -\frac{4}{2} \rfloor)x - (\lfloor \frac{1}{2} \rfloor)s = 1 - 2x. \text{ (0.1 points)}$$

That is, the Gomory cut is $y - 2x \leq 1$.

Exercise 7 0.25+0.25+0.7 points

Consider the linear program

$$\begin{aligned} \min -2x_1 - x_2 - 3x_3 \\ 2x_1 + x_2 + 2x_3 + s = 4 \\ x_1 + x_2 + x_3 + t = 2 \\ x_1, x_2, x_3, s, t \geq 0. \end{aligned}$$

To solve the linear program, we are using Dantzig-Wolfe decomposition by keeping the second constraint $x_1+x_2+x_3+t = 2$ in the master problem. That is, we have to consider only the extreme points and rays of the polyhedron $P = \{(x, s) \in \mathbb{R}^3 \times \mathbb{R} \,:\, 2x_1+x_2+2x_3+s = 4,\ x_1, x_2, x_3, s \geq 0\}$.

(a) Find all extreme points and rays of polyhedron P.

(b) Write down the corresponding explicit master problem.

(c) Solve the above linear program using Dantzig-Wolfe decomposition. Use $(x_1, x_2, x_3, s, t) = (0, 0, 0, 4, 2)$ as initial solution.

Solution:

1. Besides box constraints, there is exactly one constraint. Consequently, a basis consists of exactly one variable. The corresponding vertices are

$$v^0 = (0, 0, 0, 4)^\top, \qquad v^1 = (2, 0, 0, 0)^\top, \qquad v^2 = (0, 4, 0, 0)^\top, \qquad v^3 = (0, 0, 2, 0)^\top.$$

2. We introduce for each vertex a non-negative multiplier λ. The explicit master problem is then given by

$$\begin{aligned} \max -4\lambda_1 - 4\lambda_2 - 6\lambda_3 \\ 2\lambda_1 + 4\lambda_2 + 2\lambda_3 + s = 2, \\ \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1, \\ \lambda_0, \lambda_1, \lambda_2, \lambda_3, s \geq 0. \end{aligned}$$

3. The corresponding initial basis solution in the master problem is $(\lambda_0, s) = (1, 2)$ with basis matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. To check whether this solution is already optimal, we have to compute $(q, r)^\top = u = c_B^\top B^{-1} = (0, 0)^\top$, where c_B is the restriction of the master problem's objective to the variables in the basis. Next, we solve

$$\min\{-2x_1 - x_2 - 3x_3 - q(x_1 + x_2 + x_3) : 2x_1 + x_2 + 2x_3 + t = 4,\ x_1, x_2, x_3, t \geq 0\},$$

which has $(x_1, x_2, x_3, s) = (0, 0, 2, 0)$ as optimal solution. Since its objective value -6 is smaller than $r = 0$, the corresponding extreme point v^3 enters the basis. (0.2 points)

In the next step, we have to find a variable that leaves the basis. To this end, we execute the ratio test for the entering column $d = \binom{2}{1}$, which gives us the vector $B^{-1}d = \binom{1}{2}$. Computing the ratios for the old basis solution $(\lambda_0, s) = (1, 2)$ yields $(1, 1)$. Hence, either λ_0 or t can leave the basis. (0.2 points)

If we choose s for leaving the basis, the new basis is (λ_0, λ_3) with basis solution $(0, 1)$. The corresponding basis matrix is $B = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ with inverse $B^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$. Next, we compute

$$u = (q, r)^\top = c_B^\top B^{-1} = \frac{1}{2}(0, -6)\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} = (-3, 0)^\top.$$

As above, we solve

$$\min\{-2x_1 - x_2 - 3x_3 - q(x_1 + x_2 + x_3) : 2x_1 + x_2 + 2x_3 + t = 4,\ x_1, x_2, x_3, t \geq 0\},$$

to check whether we have already found a new solution. Due to the solution u, the problem simplifies to

$$\min\{x_1 + 2x_2 : 2x_1 + x_2 + 2x_3 + s = 4,\ x_1, x_2, x_3, s \geq 0\},$$

which has objective value $0 \geq r = 0$ in an optimal solution. Hence, we have already found an optimal solution. (0.2 points)

We can observe the same if we selected λ_0 to leave the basis. Then, the new basis is (λ_3, t) with basis solution $(1, 0)$. The corresponding basis matrix is $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ with inverse $B^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$. Next, we compute

$$u = (q, r)^\top = c_B^\top B^{-1} = (-6, 0)\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} = (0, -6)^\top.$$

As above, we solve

$$\min\{-2x_1 - x_2 - 3x_3 - q(x_1 + x_2 + x_3) : 2x_1 + x_2 + 2x_3 + s = 4,\ x_1, x_2, x_3, s \geq 0\}.$$

Since $q = 0$, an optimal solution is given by v^3, which is already contained in the basis. In particular, the objective value is $-6 \geq r$, hence we have already found an optimal solution.

In a post-processing step, we compute the optimal solution w.r.t. the original variables as

$$v^3 = (0, 0, 2, 0)^\top.\text{(0.1 points)}$$

Exercise 8 0.5+0.25+0.3+0.5+0.75 points

Let $n \geq 2$ be an integer. Consider the set

$$X = \Big\{(x, y) \in \{0,1\}^n \times \{0,1\} : y = \prod_{i=1}^{n} x_i\Big\}.$$

(a) Explain why the polyhedron P, defined by the inequalities

$$y \leq x_i, \qquad i \in \{1, \dots, n\}, \tag{1a}$$
$$\sum_{i=1}^{n} x_i \leq n - 1 + y, \tag{1b}$$
$$x_i \leq 1, \qquad i \in \{1, \dots, n\}, \tag{1c}$$
$$-x_i \leq 0, \qquad i \in \{1, \dots, n\}, \tag{1d}$$
$$y \leq 1, \tag{1e}$$
$$-y \leq 0, \tag{1f}$$

is an integer programming formulation of X, i.e., $X = P \cap (\mathbb{Z}^n \times \mathbb{Z})$.

(b) Give the definition of the dimension of a polyhedron.

(c) Show that the dimension of P is $n + 1$, in formulae, $\dim(P) = n + 1$.

(d) Prove or refute each of the following statements:
  (i) For every $i \in \{1, \dots, n\}$, Inequality (1a) defines a facet of P.
  (ii) For every $i \in \{1, \dots, n\}$, Inequality (1d) defines a facet of P.

(e) Prove that every extreme point of P is an integral vector if $n = 2$.

Solution:

(a) Let $(x, y) \in X$. Then, (x, y) is binary, and $y = 1$ if and only if $x_i = 1$ for all $i \in [n]$. (0.1 point)

The last four inequalities guarantee that any feasible integer point (x, y) is binary. (0.1 point)

The first inequality implies that y is 0 if there is at least one $i \in [n]$ with $x_i = 0$. (0.15 point)

Finally, the second inequality implies that if all entries of x are 1, then also $y = 1$, showing all properties of a point in X. (0.15 point)

(b) The dimension of a polyhedron P is the maximum number of affinely independent points in P reduced by 1. (0.25 points)

(c) We need to construct $n + 2$ affinely independent points in P.

Consider the points

- $(0, 0)$ (the null vector)
- $(e^i, 0)$ for all $i \in [n]$ (standard unit vectors in x)

- $(1, 1)$ (all-1 vector)

These points adhere to the characterization mentioned previously, i.e., all these points are contained in P. (0.1 point)

To show that these points are affinely independent, it is sufficient to show that they are linearly independent when removing the null vector.

Writing them down in a matrix, yields a matrix with determinant 1, i.e., they are linearly independent.

For this reason, the dimension of P is at least $n + 1$ (as this is a set of $n + 2$ affinely independent vectors). (0.1 point)

The dimension is at most $n + 1$ because the ambient space has dimension $n + 1$. (0.1 point)

Thus, $\dim(P) = n + 1$.

(If one of the needed points is wrong, -0.1)

(d) To show that an inequality does (not) define a facet, we have to find $n + 1$ affinely independent vectors in P satisfying the inequality with equality (or show that no such vectors exist).

For the first inequality, take

- $(0, 0)$
- $(e^j, 0)$ for all $j \in [n] \setminus \{i\}$
- $(1, 1)$

Above we have seen that they are already contained in P and affinely independent. They also satisfy the inequality with equality, hence it defines a facet. (0.25 point)

The fourth inequalities are not facet defining as they can be dominated by adding the valid inequalities $y - x_i \leq 0$ and $-y \leq 0$. (0.25 points)

(If one of the needed points is wrong, -0.1)

(e) Due to the first part of the exercise, we already know that $\text{conv}(X) \subseteq P$ and $X = P \cap (\mathbb{Z}^n \times \mathbb{Z})$. For this reason, it is sufficient to show that P is integral. (0.25 point)

To this end, we show that the constraint matrix is totally unimodular. Then, the result follows by the total unimodularity theorem since the constraint system is in inequality form, non-negativity constraints on all variables are provided, and all data is integral. (0.25 point)

Moreover, it is sufficient to show that the first two families of inequalities have a totally unimodular constraint matrix, because appending unit matrices for the upper bound constraints does not affect total unimodularity.

The constraint matrix of the first two families of inequalities has shape

$$\begin{pmatrix} -1 & & 1 \\ & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix},$$

where the last column corresponds to the y-variable.

By computing determinants of all quadratic submatrices, we see that the matrix is TU, concluding the proof. (0.25 points)