Exam Advanced Linear Programming, June 3, 2019

- Switch off your mobile phone, PDA and any other mobile device and put it far away.
- No books or other reading materials are allowed.
- This exam consists of two parts. *Write the answers to the different parts on different pieces of exam paper.* Please write down your name on every exam paper that you hand in.
- This exam consists of 5 pages containing 9 questions in total. Part 1 has 6 questions and part 2 has 3 questions.
- Answers may be provided in either Dutch or English.
- All your answers should be clearly written down, and you should provide a clear explanation. Unreadable or unclear answers may be judged as false.
- The maximum score per question is given between brackets before the question.

Good luck, veel succes!

Part 1

1 *(1 pt.)* Formulate Farkas' Lemma.

Answer.
Theorem. Given $m \times n$ matrix A and $b \in \mathbb{R}^m$, exactly one of the following two alternatives holds:
(a) $\exists\, x \geq 0 : \; Ax = b$;
(b) $\exists\, y \in \mathbb{R}^m : \; y^T A \geq 0 \wedge y^T b < 0$.

2 *(1 pt.)* Let A be an $m \times n$ matrix and let $b \in \mathbb{R}^m$. Complete the statement of the following lemma and prove it.

Lemma. *Exactly one of the following holds:*
(a) there exist $x \in \mathbb{R}^n$ *such that* $Ax \leq b$*;*
(b) ...

Answer.

(b) there exist $y \in R^m$ such that $y \geq 0$, $y^T A = 0$, and $y^T b < 0$.

Proof. We introduce slack $s \in \mathbb{R}^m$, $s \geq 0$ and we introduce non-negative variables $x^+ \geq 0$ and $x^- \geq 0$ and substitute $x = x^+ - x^-$ such that $Ax^+ - Ax^- + Is = b$ is equivalent to $\exists\, x \in \mathbb{R}^n : \; Ax \leq b$. Then we apply Farkas' Lemma to this system to yield that exactly one of the following two alternatives holds:
(a) $\exists\, x^+ \geq 0, \; x^- \geq 0, \; s \geq 0 : \; Ax^+ - Ax^- + Is = b$;
(b) $\exists\, y \in \mathbb{R}^m : \; y^T A \geq 0 \wedge -y^T A \geq 0 \wedge y^T I \geq 0 \wedge y^T b < 0$, which is equivalent to $\exists\, y \in \mathbb{R}^m, y \leq 0 : \; y^T A = 0 \wedge y^T b < 0$.
QED

3 *(1 pt.)* Given a polytope $P = Ax \geq b$ and the linear optimization problem

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in P. \end{array}$$

Prove that the set of optimal solutions of the problem is again a polytope, whose extreme points are extreme points of P.

Answer.
Proof. Let v be the optimal value of the LP, then the set of optimal solutions is given by $Q =: \{x \in P, c^T x = v\}$, which is clearly again a polytope. It remains to be proven that the extreme points of Q are extreme points of P. Suppose not. Then Q has an extreme point x such that $x = \lambda y + (1 - \lambda) z$ for $y, z \in P$ and $0 < \lambda < 1$. Since not both y and z are in Q, suppose $y \notin Q$, i.e. $c^T y > v$. This together with $c^T z \geq v$ and $c^T x = v$ yields a contradiction.
QED

4 *(0.5 + 0.5 pt.)*

(a) Consider the following linear optimization problem

$$\begin{array}{lrcrcrcrcrcrcrcrcr}
\max & Z & = & 3x_1 & - & 2x_2 & + & 4x_3 & + & 4x_4 & & & & & & & & \\
\text{s.t.} & & & 2x_1 & - & x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & & & & & = & 10 \\
& & & x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & & & = & 8 \\
& & & x_1 & - & x_2 & & & + & 2x_4 & & & & & + & x_7 & = & 4
\end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0.$$

In an iteration of the simplex method used for solving this problem we encounter a basic feasible solution with basic variables x_3, x_6 en x_1 (in that order) with the corresponding inverse basis matrix given by

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Determine if this is an optimal solution. If so, compute the solution and its value, if not perform one improving simplex step.

Answer.
The reduced objective coefficient of x_4 is $4 - c_B^T B^{-1} A_4 = 4 - \frac{10}{3} = \frac{2}{3}$. Hence positive and improvement is possible. The present solution has values $(x_3, x_6, x_1)^T = B^{-1}b = (\frac{2}{3}, \frac{10}{3}, 4)^T$. x_4 enters the basis, we compute $B^{-1}A_4 = (-\frac{2}{3}, -\frac{4}{3}, 2)^T$. Hence, x_1 must leave the basis. The new basis matrix is computed by replacing the last column of I_3 by $B^{-1}A_4$ and turn it into $(0,0,1)^T$ applying the same row operations to B^{-1}.
Any calculation errors have not been counted.

(b) Consider the simplex method applied to a standard form problem and assume that the rows of the matrix A are linearly independent. Prove that if in an iteration of the simplex method the feasible solution is moved by a positive distance then the solution value must have changed.

Answer.
Proof. Let N denote the set of non-basic indices. Let d be the improving direction of the simplex iteration. Then we have,

$$d_B = -\sum_{i \in N} B^{-1} A_i d_i,$$

$d_j = 1$ and $d_i = 0 \; \forall i \in N \backslash \{j\}$, where $j \in N$ is such that the reduced cost $\bar{c}_j$ of the variable x_j is negative.

Then we can compute the difference in cost between the point before and after the simplex iteration by:

$$\begin{aligned} c^T(x + \theta^* d) - c^T x = c^T \theta^* d = \theta^* c^T d = \theta^* \left(c_B^T d_B + \textstyle\sum_{i \in N} c_i d_i \right) \\ = \theta^* \left(\textstyle\sum_{i \in N} (c_i - c_B^T B^{-1} A_i) d_i \right) = \theta^* \left(\textstyle\sum_{i \in N} \bar{c}_i d_i \right) = \theta^* \bar{c}_j < 0 \end{aligned}$$

The last strict inequality follows because $\theta^* > 0$ and $\bar{c}_j < 0$.

QED

5 *(1 pt.)*

Given a network $\mathcal{D} = (\mathcal{N}, \mathcal{A})$. Let s be the only source node of the network and t the only sink node. Let $\delta^+(U)$ denote the subset of arcs that have their tail in U and their head in $\mathcal{N} \setminus U$ and let $\delta^-(U)$ denote the subset of arcs that have their tail in $\mathcal{N} \setminus U$ and their head in U. Let f be an s-t flow. Prove that the value of f is equal to the total capacity of $\delta^+(U)$ if and only if the following two statements hold:

$$\begin{array}{rcl} f(a) & = & 0 \quad \forall a \in \delta^-(U) \\ f(a) & = & c(a) \quad \forall a \in \delta^+(U) \end{array}$$

Answer. Proof. ($\Leftarrow$). If $f(a) = 0$ for all $a \in \delta^-(U)$ then all the flow going through the cut U from the s-side does not return to this side, hence flows into t. If $\sum_{a\in\delta^+(U)} f(a) = \sum_{a\in\delta^+(U)} c(a)$ then this flow is equal to the cut capacity.

($\Rightarrow$). If the value of f is equal to the total capacity of $\delta^+(U)$ then we know it is a maximum flow, since for any flow and any cut U we have that the value of the flow is at most the size of the cut. The rest of the proof can follow the same arguments as in the proof of the max flow min cut theorem, for which I refer to the lecture notes.

QED

6 *(1 pt)*

Flow Decomposition Theorem. *Given a network $\mathcal{D} = (\mathcal{N}, \mathcal{A})$. Let s be the only source node of the network and t the only sink node. Let f be any feasible s-t flow. There exists a set of paths $P_1, \ldots, P_K$ from s to t in the network for some K and positive scalars β_k, $k = 1, \ldots, K$, such that for every arc $a \in \mathcal{A}$ we have $f(a) = \sum_{P_k \ni a} \beta_k$. Moreover, if f is an integer flow vector, then for $k = 1, \ldots, K$, β_k can be chosen integer.*

Prove the Flow Decomposition Theorem.

Answer. Proof. The proof is similar to the proof of the flow decomposition theorem in the book and the lecture notes. Neglect the arcs in the network that do not carry any flow. Start in s and follow a single path, in each step selecting an arc with positive flow, until t is reached. Denote this path P_1, and let $\beta_1 =: \min_{a\in P_1} f(a)$. Then reduce the flow on all the arcs of P_1 by β_1. Clearly the remaining flow is still a feasible flow but the network of arcs with positive flow has lost at least one edge. Continue until the 0-flow remains.

QED

Just for your interest: In a way the above theorem tells us that we can find the maximum s-t flow in a network by just augmenting on s-t paths of the network. Moreover to find the decomposition with a minimum number of paths is an NP-hard problem.

Part 2

1 *(0.25 pt + 0.5 pt + 0.75 pt)* We are given an undirected graph $G = (V, E)$, with nodes V and edges E. Node i has a positive weight w_i. An independent set I is a subset of V such that no two edges in I are connected by an edge, i.e., each edge has at most one endpoint in I. We want to find an independent set with maximum weight.

(a) Give an integer linear programming formulation of this problem. Clearly describe the decision variables, objective function, and constraints.
Answer:
Define for each node $i \in V$ a binary variable x_i which equals 1 if i is in the independent set and 0 otherwise. We obtain the following formulation:

$$\max \sum_{i \in V} w_i x_i$$

subject to

$$x_i + x_j \leq 1 \quad \forall_{(i,j) \in E}$$
$$x_i \in \{0, 1\} \quad \forall_{i \in V}$$

(b) A clique C is a subset of V such that every pair of nodes in C is connected by an edge. Define a valid inequality for the independent set polyhedron based on a clique C.
Answer:

$$\sum_{i \in C} x_i \leq 1.$$

(c) A clique C is called maximal if it is impossible to obtain a clique by adding a node C. Prove that for maximal cliques, the inequalities from part (b) are facet inducing. You may assume that the independent set polyhedron is full-dimensional.
Answer:
Let n be the number of nodes and let P be the independent set polyhedron, i.e. the convex hull of the set of integer solutions to the independent set problem. Let C be a maximal clique. We have to prove that $\{x \in P | \sum_{i \in C} x_i = 1\}$ has dimension $n - 1$, i.e. there are n affinely independent vectors in $\{x \in P | \sum_{i \in C} x_i = 1\}$. These vectors are:

- The unit vectors $\bar{y}_i = e_i$ for $i \in C$.
- Let $i \notin C$. Since C is a maximal clique there is at least one $j \in C$ such that $(i, j) \notin E$. Now $\bar{y}_i = e_i + e_j$ is a feasible solution.

It is easy to see that the only solution of $\sum_{j \in V} \lambda_j \bar{y}_j = 0$ and $\sum_{j \in V} \lambda_j \bar{y}_j = 0$ is $\lambda_j = 0$ for all j.

□

2 *(0.5 pt. + 0.75 pt.)* We consider the following vehicle routing problem. There are m vehicles located at a central depot. The vehicles have to transport goods to n customers. The distance from customer i to customer j equals d_{ij}. Observe that it is possible that $d_{ij} \neq d_{ji}$, e.g. because of one-way streets in a city. However, we assume that the triangle inequality holds for the distances. Moreover, the distance from the depot to customer j equals d_{0j} and from customer j to the depot d_{j0}. We assume that the capacity of the vehicles is not a limiting factor, so you do not have to take it into account. The question is to find routes for the m vehicles such that the total driving distance is minimal.

(a) The vehicle routing problem can be formulated by using vehicle plans; a vehicle plan describes a route along a subset of the customers driven by one vehicle. Give an integer linear programming formulation for this problem based on routes.

Answer:

Let R be collection of ordered subsets of $\{1, \ldots, n\}$. Each $r \in R$ defines a vehicle plan (route). For a given route r we define a binary decision variable x_r which equals 1 if route r is selected and 0 otherwise. Let d_r be the total length of the route. Moreover we define the indicator a_{jr} equal to 1 if customer j is in route r and 0 otherwise. Now we obtain the following formulation.

$$\min \sum_{r \in R} d_r x_r$$

subject to

$$\sum_{r \in R} a_{jr} x_r = 1 \quad \forall_j \tag{1}$$

$$\sum_{r \in R} x_r = m \tag{2}$$

$$x_r \in \{0, 1\} \quad \forall_{r \in R} \tag{3}$$

The first constraint ensures that each customers is visited exactly once, and the second constraint enforces that we select m routes.

(b) Describe how the LP-relaxation of this formulation can be solved by column generation. Your description should include a formulation of the pricing problem. You do not need to describe how to solve the pricing problem.

Answer:

1. Start with a restricted master LP which contains a small subset of R^* of vehicle plans. R^* can for examplebe obtained by taking as route 1 customers $1, \ldots, \lfloor \frac{n}{m} \rfloor$, as route 2 customers $\lfloor \frac{n}{m} \rfloor + 1, \ldots, 2\lfloor \frac{n}{m} \rfloor$ etc. (make sure that route m includes customer n).
2. Solve the restricted master LP.

3. Solve the pricing problem to find out if there are vehicle plans outside the restricted master problem with negative reduced cost. If yes, add the variable corresponding to this vehicle plan to the restricted master problem and go to step 2. If no, the LP has been solved to optimality.

In the pricing problem we have to minimize the reduced cost. Suppose we have solved the restricted master problem. Let π_j $(j = 1, \ldots, n)$ be the dual variables for the first constraints and μ the dual variable f the second constraints. The reduced cost of route r are

$$\sum_{(i,j)\in r_{op}} d_{ij} - \sum_{j\in r} \pi_j - \mu,$$

where r_{op} denotes the sts of pairs of customers that are visited consecutively in route r. This can be rewritten as:

$$\sum_{(i,j)\in r_{op}} (d_{ij} - \pi_i) - \mu.$$

So we have to solve a variant of the travelling salesmen problem in which we are allowed to visit only a subset of the customers. Note that distances may be negative.

3 *(0.5 pt. + 0.25 pt. + 0.5 pt.)* The company Fiber & Co wants to construct an optical fibre network in a medium-sized city. Finding the best network design boils down to the problem MINIMUM SPANNING TREE WITH UNREALIABLE EDGES. We are given an undirected graph $G = (V, E)$. Each edge $e \in E$ has cost c_e. A tree is a connected subgraph without cycles. A tree is a *spanning tree* of a graph if it contains all nodes. We want to find a spanning tree with minimum cost. However, there is a subset W of the edges, which are unreliable because they break down regularly. For this reason, we want to select at most q edges from the set W. We assume that $|W| > q$.

(a) Give an integer linear programming formulation for this problem. Clearly describe the decision variables, objective, and constraints.

Answer:

Let $n = |V|$. Define for each edge $e \in E$ a binary variable x_e which equals 1 if e is in the spanning tree and 0 otherwise. We obtain the following formulation:

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\sum_{e \in E} x_e = n - 1 \tag{4}$$

$$\sum_{e \subset S} x_e \leq |S| - 1 \quad \forall_{S \subset V, S \neq \emptyset, N} \tag{5}$$

$$\sum_{e \in W} x_e \leq q \tag{6}$$

$$x_e \in \{0, 1\} \quad \forall_{e \in E} \tag{7}$$

(b) Write the Lagrangean relaxation subproblem that you obtain by dualizing the constraint on the maximum number of unreliable edges. If you did not find an answer in part (a) you can use the formulation $\min\{cx | Ax \leq b,\ dx \leq f,\ x \in \{0, 1\}^n\}$, where $dx \leq f$ models the maximum number of unreliable edges.

Answer:

Let $\lambda > 0$.

$$\min \sum_{e \in E} c_e x_e - \lambda(q - \sum_{e \in W} x_e)$$

subject to

$$\sum_{e \in E} x_e = n - 1 \tag{8}$$

$$\sum_{e \subset S} x_e \leq |S| - 1 \quad \forall_{S \subset V, S \neq \emptyset, N} \tag{9}$$

$$x_e \in \{0, 1\} \quad \forall_{e \in E} \tag{10}$$

(c) Let Z_{IP} be the optimal value of the original ILP formulation and $Z(\lambda)$ be the optimal value of the Lagrangean subproblem with multiplier λ. Do we have $Z(\lambda) \leq Z_{IP}$ or $Z(\lambda) \geq Z_{IP}$? Prove your answer.

Answer:

We have $Z(\lambda) \leq Z_{IP}$. Let x^* be the optimal solution of the IP. Since x^* is feasible we have $\sum_{e \in W} x^*_e \, leqq$. Consequently $Z_\lambda \leq \sum_{e \in E} c_e x^*_e - \lambda(q - \sum_{e \in W} x^*_e) \leq \sum_{e \in E} c_e x^*_e = Z_{IP}$.

□